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Optimal Asset-Liability Management

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Optimal Asset-Liability Management

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Abstract

In this thesis, Mean-Variance Asset-Liability management is studied in a multi-period setting. An investor aims at finding an optimal investment strategy in order to maximise the mean-variance objective. The prices of assets and liabilities are formulated as geometric Brownian motions and we further extend them to exponential Levy process. By the Bellman principle, the explicit optimal solution is obtained under backward induction.

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1 Introduction

Liabilities have a significant impact on portfolio selection problems. In this thesis, we examine classical results on portfolio selection without liabilities and then extend the model to include liabilities.

The key difference between an asset and a liability from the point of view of this thesis is that assets are assumed to be tradable, whereas liabilities are not tradable. Thus an investor may rebalance the asset side of their portfolio in response to market changes, but the liability side of their portfolio remains unchanged.

In this thesis, we focus on the study of Mean-Variance Asset-Liability management over multiple periods. We consider the wealth process as the surplus of total liabilities from total assets. We assume that the prices of assets and liabilities are governed by geometric Brownian motions and then we further formulate the assets and liabilities dynamics with jump processes. We consider investor who aims at maximising the expected terminal wealth, but in the meanwhile minimising the risk measured by the variance. Under this mean-variance setting, we first study the optimal investment strategy in a single period model. The backward induction is used to drive the explicit solution for the multi-period case. By using the backward induction, we obtain an analytic solution of the optimal investment strategy. The optimal result is shown to be affine in current wealth and current liability price. All optimal coefficients can be obtained by backward recursions. Finally, in the numerical studies, we illustrate and compare cases with different parameters.

The outline of this thesis is as follows. Chapter 2 gives an overview of the existing literatures. Section 2.1 and section 2.2 present the basic framework of the Markowitz theory [15] and Merton's problem [16].

In chapter 3, the optimal strategy of Asset-Liability portfolio selection problem is investigated under the mean-variance criteria. It is the central result in this thesis. In a single period setting, we find the explicit investment strategies when assets and liabilities follow a general Gaussian model. Then in a multiple period setting, we find the explicit investment strategies for a single geometric Brownian motion asset and multiple geometric Brownian motion liabilities.

In chapter 4, we further extend the model using jump process which gives a more realistic market model.

Chapter 5 records down numerical results. All our analytic results are verified numerically. We shows that multi-period strategy has a significantly better outcome than the single period strategy. We all consider sensitivities to different parameters, in particular the liabilities and jumps.

2 Literature Review

The development of Asset Liability Management can be traced back to the 1950's when Markowitz introduced Modern Portfolio Theory [15]. He obtained the optimal asset allocation that provides the highest expected return for a given level of risk measured by standard deviation. Later on, Samuelson (1969) [19] and Merton (1971) [17] extend Markowitz's work to multi-period and continuous cases via formulating it as a stochastic process, corresponding to life-time planning of consumption and investment decision. The mean-variance framework has since been used in various studies of portfolio selections.

2.1 Markowitz Model

In the following sections, we shall describe the Markowitz model of optimal investment. This is a single period multi asset model. A fundamental notion, the efficient frontier, is also introduced. Our discussions mainly focus on the Markowitz model [15].

2.1.1 Background

An investor, starting with an amount of cash, seeks for an efficient investment portfolio in order to have a great return in the future. Investing in a bond has low risk but may result in a low income compared with other investments. As an investor, one would like to make high profit meanwhile lowering the risk of losing. Thus, one wonders how to balance budget among various assets in order to have the maximum expected return in the future for a fixed risk.

In 1952, Markowitz published the Portfolio Theory [15] in which the Mean-Variance Portfolio is proposed as a quantitative framework to reflect trading off one's expected return against risk. In the Markowitz setting, risk is modelled in terms of variance of the return. The fundamental objective is to choose the optimal investment to lower the risk at a fixed return or equivalently find the greatest return under a specific risk.

2.1.2 Basic model

The Markowitz model is based on a single period setting. Suppose we invest X_0 amount of money in an asset at time $t = 0$ and after a period of time, we sell the asset at price X_1 . Here, we denote the ratio

$$R = \frac{X_1}{X_0}$$

as the return on the asset. Then, the rate of return on the asset is defined as

$$\mu = \frac{X_1 - X_0}{X_0} = R - 1.$$

Thus, we have

$$X_1 = RX_0 = (1 + \mu)X_0.$$

Now consider the case with n securities. Let X_0 be the initial amount of money we are holding at time $t = 0$. We wish to distribute this amount of money into n asset and the amount that we assign to asset i is $X_{0i} = u_i X_0$ where u_i denotes the fraction of investment in asset i , so $\sum_{i=1}^n u_i = 1$. Let the total return on asset i be

R_i and then the payoff from this portfolio after a period of time is

$$X_1 = \sum_{i=1}^n R_i u_i X_0 = X_0 \sum_{i=1}^n R_i u_i$$

so the total return R is

$$R = \sum_{i=1}^n R_i u_i.$$

In addition, we have the rate of return on each asset i to be $\mu_i = R_i - 1$, and thus the rate of return on the portfolio is

$$\mu = \sum_{i=1}^n \mu_i u_i.$$

2.1.3 Problem setting

The Markowitz model selects investment proportions (u_1, u_2, \dots, u_n) in various assets in order to minimise the risk for a given amount of return, or maximise the expected return for a specific level of risk.

Let R be the vector of returns of all assets, i.e.,

$$\mathbf{R} = \begin{pmatrix} R_1 \\ R_2 \\ \vdots \\ R_n \end{pmatrix}.$$

In the Markowitz model, the return R_i is formulated as random variables for asset i . Let $\xi_i = \mathbb{E}(R_i)$ be the expected value of R_i and define the expected return vector as

$$\boldsymbol{\xi} = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{pmatrix}.$$

Set $\text{cov}(R_i, R_j) = \mathbb{E}(R_i R_j) - \xi_i \xi_j$ as the covariance of return R and the variance-

covariance matrix of R is given by

$$\mathbf{\Omega} = \begin{pmatrix} \text{Var}(R_1) & \cdots & \text{cov}(R_1, R_n) \\ \vdots & \ddots & \vdots \\ \text{cov}(R_n, R_1) & \cdots & \text{Var}(R_n) \end{pmatrix}.$$

If $\mathbf{u} = (u_1, u_2, \dots, u_n)^T$ is a set of weights assigned to the portfolio, then the rate of return $\mu = R_i - 1 = \sum_{i=1}^n \mu_i u_i$ on this portfolio is a random variable with mean $\boldsymbol{\xi}^T \mathbf{u}$ and variance $\mathbf{u}^T \mathbf{\Omega} \mathbf{u}$.

We seek to find an investment strategy (u_1, u_2, \dots, u_n) such that the variance Ω is minimised under some constraints. With these settings, the Markowitz problem becomes:

$$\begin{aligned} & \text{minimising} \quad \mathbf{u}^T \mathbf{\Omega} \mathbf{u} \\ & \text{s.t.} \quad \mathbf{1}^T \mathbf{u} = 1 \quad , \\ & \quad \quad \boldsymbol{\xi}^T \mathbf{u} = R_p \end{aligned} \tag{2.1}$$

where R_p is the desired return on the portfolio and $\mathbf{1}$ is a vector of ones.

2.1.4 Solution to the Markowitz Problem (A Lagrangian Approach)

The Lagrangian of the Markowitz problem can be written as:

$$\mathcal{L} = \mathbf{u}^T \mathbf{\Omega} \mathbf{u} + \lambda_1 (1 - \mathbf{1}^T \mathbf{u}) + \lambda_2 (R_p - \boldsymbol{\xi}^T \mathbf{u}),$$

where λ_1 and λ_2 are the Lagrangian multipliers for the constraints (2.1).

From the first-order conditions, we then differentiate \mathcal{L} with respect to u , λ_1 and

λ_2 respectively and set the derivative to zero.

$$\begin{aligned}\nabla_{\mathbf{u}}\mathcal{L} = 0 &\implies 2\mathbf{\Omega}\mathbf{u} - \lambda_1\mathbf{1} - \lambda_2\boldsymbol{\xi} = \mathbf{0} \\ &\implies \mathbf{u} = \frac{1}{2}\mathbf{\Omega}^{-1}(\mathbf{1} \ \boldsymbol{\xi}) \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix};\end{aligned}\tag{2.2}$$

$$\frac{\partial\mathcal{L}}{\partial\lambda_1} = 0 \implies \mathbf{1}^T\mathbf{u} = 1;\tag{2.3}$$

$$\frac{\partial\mathcal{L}}{\partial\lambda_2} = 0 \implies \boldsymbol{\xi}^T\mathbf{u} = R_p.\tag{2.4}$$

To determine λ_1 and λ_2 , we obtain from the two constraints (2.3) and (2.4) that

$$\begin{aligned}1 &= \mathbf{1}^T\mathbf{u} \\ &= \mathbf{1}^T\left(\frac{1}{2}\mathbf{\Omega}^{-1}(\mathbf{1} \ \boldsymbol{\xi}) \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}\right) \\ &= \frac{1}{2}\lambda_1\mathbf{1}^T\mathbf{\Omega}^{-1}\mathbf{1} + \frac{1}{2}\lambda_2\mathbf{1}^T\mathbf{\Omega}^{-1}\boldsymbol{\xi}.\end{aligned}\tag{2.5}$$

$$\begin{aligned}R_p &= \boldsymbol{\xi}^T\mathbf{u} \\ &= \boldsymbol{\xi}^T\left(\frac{1}{2}\mathbf{\Omega}^{-1}(\mathbf{1} \ \boldsymbol{\xi}) \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}\right) \\ &= \frac{1}{2}\lambda_1\boldsymbol{\xi}^T\mathbf{\Omega}^{-1}\mathbf{1} + \frac{1}{2}\lambda_2\boldsymbol{\xi}^T\mathbf{\Omega}^{-1}\boldsymbol{\xi}.\end{aligned}\tag{2.6}$$

From (2.5) and (2.6), we can solve for λ_1 and λ_2 :

$$\frac{1}{2}\begin{pmatrix} \mathbf{1}^T\mathbf{\Omega}^{-1}\mathbf{1} & \mathbf{1}^T\mathbf{\Omega}^{-1}\boldsymbol{\xi} \\ \boldsymbol{\xi}^T\mathbf{\Omega}^{-1}\mathbf{1} & \boldsymbol{\xi}^T\mathbf{\Omega}^{-1}\boldsymbol{\xi} \end{pmatrix}\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 1 \\ R_p \end{pmatrix}.$$

Put

$$\mathbf{A} = \begin{pmatrix} \mathbf{1}^T\mathbf{\Omega}^{-1}\mathbf{1} & \mathbf{1}^T\mathbf{\Omega}^{-1}\boldsymbol{\xi} \\ \boldsymbol{\xi}^T\mathbf{\Omega}^{-1}\mathbf{1} & \boldsymbol{\xi}^T\mathbf{\Omega}^{-1}\boldsymbol{\xi} \end{pmatrix} = \begin{pmatrix} \mathbf{1}^T \\ \boldsymbol{\xi}^T \end{pmatrix} \mathbf{\Omega}^{-1}(\mathbf{1} \ \boldsymbol{\xi}),$$

We have

$$2\mathbf{A}^{-1}\begin{pmatrix} 1 \\ R_p \end{pmatrix} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}.$$

Substitute into equation (2.2), we can obtain the optimal solution as:

$$\mathbf{u} = \boldsymbol{\Omega}^{-1} \begin{pmatrix} \mathbf{1} & \boldsymbol{\xi} \end{pmatrix} \mathbf{A}^{-1} \begin{pmatrix} 1 \\ R_p \end{pmatrix}.$$

Then we can calculate the portfolio variance σ^2 for a given R_p by:

$$\begin{aligned} \sigma^2 &= \mathbf{u}^T \boldsymbol{\Omega} \mathbf{u} \\ &= \left(\boldsymbol{\Omega}^{-1} \begin{pmatrix} \mathbf{1} & \boldsymbol{\xi} \end{pmatrix} \mathbf{A}^{-1} \begin{pmatrix} 1 \\ R_p \end{pmatrix} \right)^T \cdot \boldsymbol{\Omega} \cdot \left(\boldsymbol{\Omega}^{-1} \begin{pmatrix} \mathbf{1} & \boldsymbol{\xi} \end{pmatrix} \mathbf{A}^{-1} \begin{pmatrix} 1 \\ R_p \end{pmatrix} \right) \\ &= \begin{pmatrix} \mathbf{1} & R_p \end{pmatrix} (\mathbf{A}^{-1})^T \begin{pmatrix} \mathbf{1}^T \\ \boldsymbol{\xi}^T \end{pmatrix} (\boldsymbol{\Omega}^{-1})^T \boldsymbol{\Omega} \boldsymbol{\Omega}^{-1} \begin{pmatrix} \mathbf{1} & \boldsymbol{\xi} \end{pmatrix} \mathbf{A}^{-1} \begin{pmatrix} 1 \\ R_p \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{1} & R_p \end{pmatrix} (\mathbf{A}^{-1})^T \mathbf{A}^T \mathbf{A}^{-1} \begin{pmatrix} 1 \\ R_p \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{1} & R_p \end{pmatrix} \mathbf{A}^{-1} \begin{pmatrix} 1 \\ R_p \end{pmatrix}. \end{aligned}$$

Thus, the frontier $\sigma^2 - R_p$ is a hyperbola.

Figure 2.1.4.1 shows an example of the $\sigma^2 - R_p$ frontier. We exhibited data of ten financial assets from the market. The blue circles shows the distribution of ten individual assets according to their means of returns and standard deviations. The black curve depicts the entire investment opportunity set, which is the combination of all possible portfolios. This reveals the risk-return combination of all portfolios formed by ten assets. The square is the global minimum variance portfolio, which entails the lowest risk among all the portfolio on the investment opportunity set.

2.1.5 The Efficient Frontier

From the Markowitz model, investor will get different combination of return R_p and risk σ^2 for different choices of investment allocations. All those pairs of (σ^2, R_p)

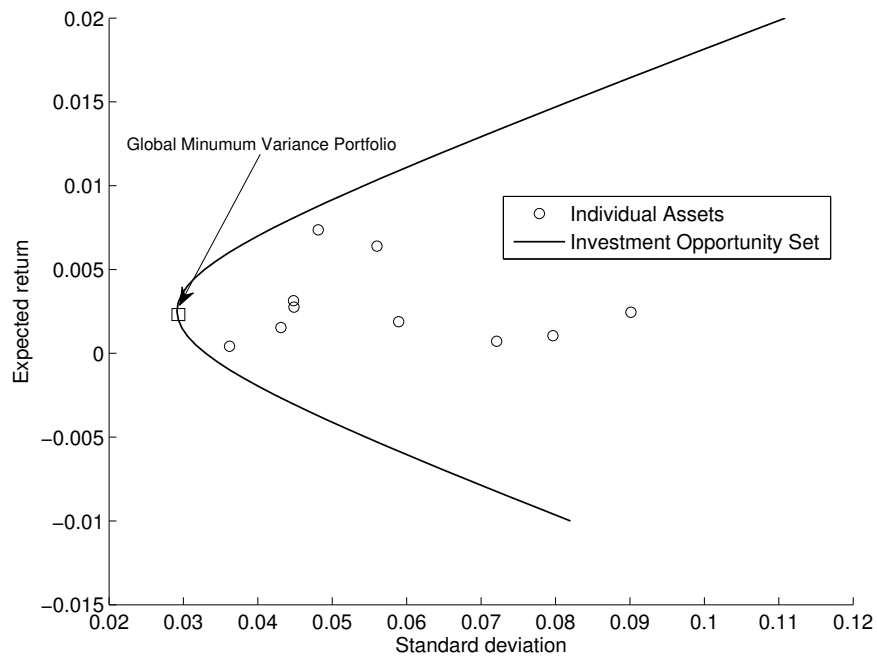


Figure 2.1.4.1: Investment Opportunity: The circles denote the return-risk distribution of ten individual assets; The black curve denotes investment opportunity set that includes all possible portfolio that one can invest in.

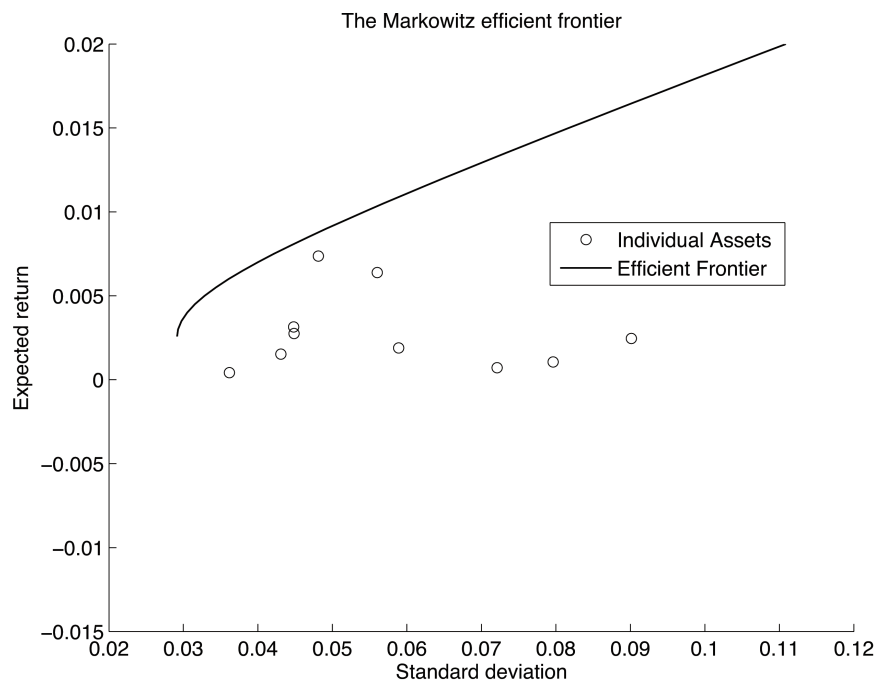


Figure 2.1.5.1: The Markowitz Efficient Frontier: The curve depicts the Markowitz Efficient Frontier, the optimal combination of risk and return.

together yield the lowest risk for a given return from the efficient frontier [15]. The efficient frontier always give an optimal portfolio that offers the highest expected return for a specific risk. Those portfolios lie below the efficient frontier are sub-optimal because they do not provide enough return for the same level of risk.

An investor is assume to be risk averse i.e. when given two portfolios with the same expected return, one may alway decide to choose the investment with a lower risk. Hence, a risk-averse investor is always willing to choose a portfolio on the efficient frontier. Figure 2.1.5.1 singles out the Markowitz efficient frontier for the ten assets case from figure 2.1.4.1. The Markowitz setting leads to the efficient frontier as a curve starting from the minimum variance portfolio. The following theorem shows that any efficient portfolio can be deduced by two of the mutual portfolio lies on the efficient frontier.

2.1.6 Two Fund Theorem

Theorem 2.1.6.1. *Any investor's optimal portfolio can be constructed by holding two mutual funds in certain appropriate ratio $\omega \in [0, 1]$. In fact, let u_a, u_b be two mutual fund portfolios with mean R_a and R_b respectively. Then any portfolio $u_c = \omega u_a + (1 - \omega)u_b$ is also an optimal portfolio on the efficient frontier i.e.*

$$\mathbf{u}_c = \mathbf{\Omega}^{-1} \begin{pmatrix} \mathbf{1} & \boldsymbol{\xi} \end{pmatrix} \mathbf{A}^{-1} \begin{pmatrix} 1 \\ R_c \end{pmatrix}$$

where

$$R_c = \mathbb{E}[\omega u_a + (1 - \omega)u_b] = \omega R_a + (1 - \omega)R_b.$$

Proof of Theorem 2.1.6.1:

Firstly, we would like to prove that u_c is a portfolio on the efficient frontier.

Since portfolio \mathbf{u}_c is constructed from portfolio \mathbf{u}_a and \mathbf{u}_b with ratio ω , we can obtain

$$\begin{aligned}
\mathbf{u}_c &= \omega \mathbf{u}_a + (1 - \omega) \mathbf{u}_b \\
&= \omega \boldsymbol{\Omega}^{-1} \begin{pmatrix} \mathbf{1} & \boldsymbol{\xi} \end{pmatrix} A^{-1} \begin{pmatrix} 1 \\ R_a \end{pmatrix} + (1 - \omega) \boldsymbol{\Omega}^{-1} \begin{pmatrix} \mathbf{1} & \boldsymbol{\xi} \end{pmatrix} A^{-1} \begin{pmatrix} 1 \\ R_b \end{pmatrix} \\
&= \boldsymbol{\Omega}^{-1} \begin{pmatrix} \mathbf{1} & \boldsymbol{\xi} \end{pmatrix} A^{-1} \begin{pmatrix} \omega R_a + (1 - \omega) R_b \end{pmatrix} \\
&= \boldsymbol{\Omega}^{-1} \begin{pmatrix} \mathbf{1} & \boldsymbol{\xi} \end{pmatrix} A^{-1} \begin{pmatrix} 1 \\ R_c \end{pmatrix}
\end{aligned}$$

where $R_c = \omega R_a + (1 - \omega) R_b$. Thus \mathbf{u}_c is a mutual portfolio.

Secondly, we would like to show that any portfolio \mathbf{u} can be written as a combination of two funds.

Let

$$R_c = \omega R_a + (1 - \omega) R_b$$

with

$$\omega = \frac{R_c - R_b}{R_a - R_b}.$$

Portfolio \mathbf{u}_c can be written as

$$\begin{aligned}
\mathbf{u}_c &= \boldsymbol{\Omega}^{-1} \begin{pmatrix} \mathbf{1} & \boldsymbol{\xi} \end{pmatrix} A^{-1} \begin{pmatrix} 1 \\ R_c \end{pmatrix} \\
&= \boldsymbol{\Omega}^{-1} \begin{pmatrix} \mathbf{1} & \boldsymbol{\xi} \end{pmatrix} A^{-1} \begin{pmatrix} \omega R_a + (1 - \omega) R_b \end{pmatrix} \\
&= \omega \boldsymbol{\Omega}^{-1} \begin{pmatrix} \mathbf{1} & \boldsymbol{\xi} \end{pmatrix} A^{-1} \begin{pmatrix} 1 \\ R_a \end{pmatrix} + (1 - \omega) \boldsymbol{\Omega}^{-1} \begin{pmatrix} \mathbf{1} & \boldsymbol{\xi} \end{pmatrix} A^{-1} \begin{pmatrix} 1 \\ R_b \end{pmatrix} \\
&= \omega \mathbf{u}_a + (1 - \omega) \mathbf{u}_b.
\end{aligned}$$

Thus, \mathbf{u}_c can be written as a combination of two portfolio \mathbf{u}_a and \mathbf{u}_b with mean R_a and R_b respectively.

We finished the proof of Two Fund Theorem.

2.1.7 One Fund Theorem

If a risk-free asset is taken into account, we have the One Fund Theorem.

Theorem 2.1.7.1. *Any efficient portfolio can be constructed as a linear combination of the risk-free asset and the market portfolio. The market portfolio mentioned here refers to holding a proportion of all risky assets in the market.*

Suppose that a risk-free asset is available with mean return R_f . Let $\mathbf{u} = (u_1, u_2, \dots, u_n)^T$ be a set of weights assigned to the risky assets, then $1 - \mathbf{1}^T \mathbf{u}$ is assigned to the riskless asset. Consider the new portfolio with mean

$$R_p = \boldsymbol{\xi}^T \mathbf{u} + (1 - \mathbf{1}^T \mathbf{u})R_f$$

and variance

$$\sigma_p^2 = \mathbf{u}^T \boldsymbol{\Omega} \mathbf{u}$$

since the covariance between the risk free asset and the risky asset is zero.

In this case, the efficient sets representing (σ_p^2, R_p) lies on a tangent line touches the original frontier (with risky assets only) at point F, joining $(0, R_f)$ and F . F is a specific single fund of risky assets. If lending and borrowing from a risk free asset is allowed, the straight line can possibly extend to the right side of point F up to infinity.

The One Fund Theorem states that there exists such a single fund F of risky assets such that any portfolio can be constructed as a combination of the fund F and the risk free asset.

Then the Markowitz efficient frontier is a straight line with equation:

$$R_p = \mathbf{1}^T \boldsymbol{\Omega}^{-1} (\boldsymbol{\xi} - R_f \mathbf{1}) \sigma_p^2 + R_f.$$

Proof of Theorem 2.1.7.1:

Considering a new portfolio with a risk free asset and the proportion of investing in a risky asset is u and $1 - \mathbf{1}^T \mathbf{u}$ in risk free assets. Then the problem becomes

$$\begin{aligned} \text{minimising} \quad & \mathbf{u}^T \boldsymbol{\Omega} \mathbf{u} \\ \text{s.t.} \quad & \boldsymbol{\xi}^T \mathbf{u} + (1 - \mathbf{1}^T \mathbf{u}) R_f = R_p. \end{aligned} \tag{2.7}$$

Put the Lagrangian function as

$$\mathcal{L} = \mathbf{u}^T \boldsymbol{\Omega} \mathbf{u} - \lambda (R_p - \boldsymbol{\xi}^T \mathbf{u} - (1 - \mathbf{1}^T \mathbf{u}) R_f),$$

where λ is the Lagrange multipliers for the constraints (2.7). From the first-order conditions, we differentiate \mathcal{L} with respect to u and λ respectively and set the derivative to zero.

$$\begin{aligned} \nabla_{\mathbf{u}} \mathcal{L} = 0 & \implies 2\boldsymbol{\Omega} \mathbf{u} + \lambda \boldsymbol{\xi} - \lambda R_f \mathbf{1} = 0 \\ & \implies \mathbf{u} = \frac{1}{2} \lambda \boldsymbol{\Omega}^{-1} (R_f \mathbf{1} - \boldsymbol{\xi}); \end{aligned} \tag{2.8}$$

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \lambda} = 0 & \implies R_p - \boldsymbol{\xi}^T \mathbf{u} - (1 - \mathbf{1}^T \mathbf{u}) R_f = 0 \\ & \implies R_f - R_p = (R_f \mathbf{1} - \boldsymbol{\xi})^T \mathbf{u}. \end{aligned} \tag{2.9}$$

From (2.8) and (2.9), we can determine λ from

$$\frac{\lambda}{2} = \frac{R_f - R_p}{(R_f \mathbf{1} - \boldsymbol{\xi})^T \boldsymbol{\Omega}^{-1} (R_f \mathbf{1} - \boldsymbol{\xi})}.$$

Substitute into (2.8), we have

$$\mathbf{u} = \frac{(R_f - R_p) \boldsymbol{\Omega}^{-1} (R_f \mathbf{1} - \boldsymbol{\xi})}{(R_f \mathbf{1} - \boldsymbol{\xi})^T \boldsymbol{\Omega}^{-1} (R_f \mathbf{1} - \boldsymbol{\xi})}.$$

To obtain σ_p^2 , we substitute (2.8) into (2.7)

$$\begin{aligned} \sigma_p^2 &= \mathbf{u}^T \boldsymbol{\Omega} \mathbf{u} \\ &= \mathbf{u}^T \boldsymbol{\Omega} \frac{1}{2} \lambda \boldsymbol{\Omega}^{-1} (R_f \mathbf{1} - \boldsymbol{\xi}) \\ &= \frac{1}{2} \lambda \mathbf{u}^T \boldsymbol{\Omega} \boldsymbol{\Omega}^{-1} (R_f \mathbf{1} - \boldsymbol{\xi}) \\ &= \frac{1}{2} \lambda \mathbf{u}^T (R_f \mathbf{1} - \boldsymbol{\xi}) \\ &= \frac{1}{2} \lambda ((R_f \mathbf{1} - \boldsymbol{\xi})^T \mathbf{u})^T \\ &= \frac{1}{2} \lambda (R_f - R_p)^T \\ &= \frac{(R_f - R_p)(R_f - R_p)^T}{(R_f \mathbf{1} - \boldsymbol{\xi})^T \boldsymbol{\Omega}^{-1} (R_f \mathbf{1} - \boldsymbol{\xi})} \\ &= \frac{(R_f \mathbf{1} - R_p)}{\mathbf{1}^T \boldsymbol{\Omega}^{-1} (R_f \mathbf{1} - \boldsymbol{\xi})}. \end{aligned}$$

Thus, we have

$$R_p = \mathbf{1}^T \boldsymbol{\Omega}^{-1} (\boldsymbol{\xi} - R_f \mathbf{1}) \sigma_p^2 + R_f,$$

which is a line with slope $\mathbf{1}^T \boldsymbol{\Omega}^{-1} (\boldsymbol{\xi} - R_f \mathbf{1})$.

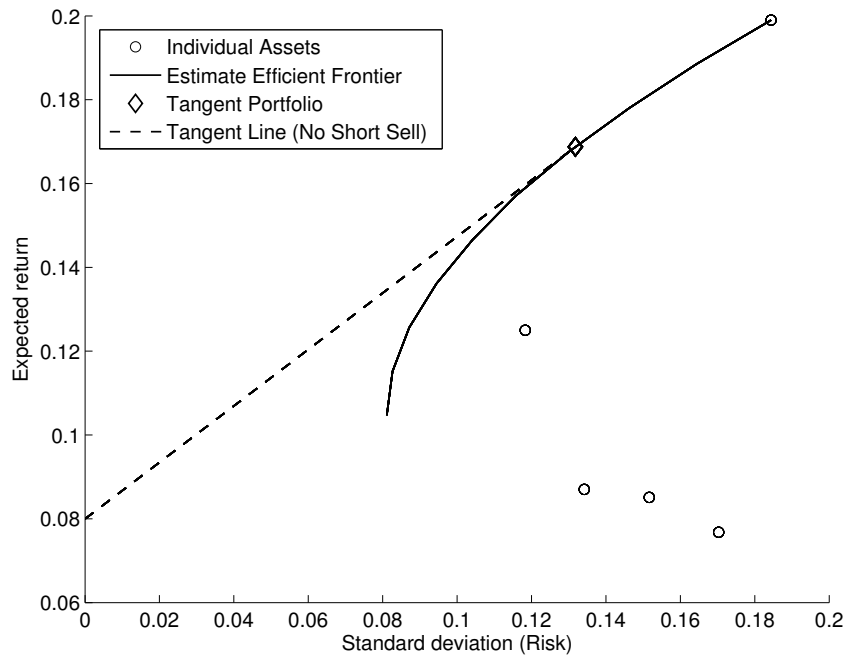


Figure 2.1.7.1: Tangent Portfolio

Figure 2.1.7.1 illustrates the efficient frontier with tangent line in a no short sell case. Given five risky assets from the market and a riskless asset with risk-free rate 8%, we plot a tangent portfolio which are combinations of cash and risky assets. The tangent portfolio with a diamond marker is the tangent point to the original efficient frontier, representing the portfolio of holding risky assets only.

2.1.8 Sharpe Ratio

Another popular way to determine an optimal portfolio is to maximise the Sharpe ratio [20]. The Sharpe ratio is a measurement of return to risk that represents the expected return over per unit of risk. It is defined as

$$\mathcal{S} = \frac{R_p - R_f}{\sigma},$$

where R_p is the expected asset return, R_f is the return on a risk-free asset and σ is the standard deviation of the asset.

A portfolio with maximum Sharpe ratio is the same as the tangent portfolio from the mutual fund theorem in 2.1.7.1. In fact, the slope of the tangent portfolio is the Sharpe ratio.

In summary, we introduce the Markowitz model and the derivation of the efficient frontier in this section. Figure 2.1.4.1 plots all possible portfolios of risky assets in a risk return region. Figure 2.1.5.1 shows the efficient frontier which is the line along the upper edge of this region. We then introduce the One Fund Theorem involving a risk free asset. Figure 2.1.7.1 shows the combination of risky portfolios and a risk free asset. Finally, the tangent portfolio is the same as a portfolio maximising the Sharpe ratio.

2.2 Merton's Problem

In section 2.1, we discussed the Markowitz Model, a one-period investment problem. By contrast, Merton's problem solves the portfolio selection problem in continuous time. Our discussion below mainly uses on the setting from Merton's problem [16] and a result on exponential hedging [10].

2.2.1 Merton's Setting

Merton's problem [16] assumes that an investor starts with initial wealth X_0 and at any time t , the wealth process follows the following stochastic differential equation

$$dX_t = \left[\left(r + \pi_t(\mu - r) \right) X_t - c_t \right] dt + X_t \pi_t \sigma dW_t, \quad (2.10)$$

where r is the risk free interest rate, μ and σ are the expected return and volatility of the risky asset and W_t denotes a standard Brownian motion. Investors are able to choose the consumption of wealth c_t and investment proportion in stock π_t . Their objective is to choose a proper control (π, c) to maximise the expected value of a utility function U . The objective is to maximise the accumulated consumption and terminal wealth, i.e.

$$\text{maximising } \mathbb{E} \left[\int_0^T U_1(t, c_t) dt + U_2(X_T) \right].$$

Two particular cases are the infinite horizon problem

$$\text{maximising } \mathbb{E} \left[\int_0^\infty U(t, c_t) dt \right]$$

and the terminal wealth problem

$$\text{maximising } \mathbb{E} [U(X_T)].$$

There are two main approaches to solve Merton's problem [18]. First is the stochastic control approach [17]. Merton derives the Bellman equation for the value function. By solving the Hamilton-Jacobi-Bellman equations, the result can be calculated in terms of the derivatives of the utility function. Although the analytic form of these derivatives is rarely known, one can turn to numerical methods in solving the PDEs. Thanks to the well-developed PDE techniques and numerical methods, explicit solutions to some special cases of utility functions have already been derived in [1][24].

Another approach is based on the martingale theory [3][7]. The central idea here is to evaluate suboptimal dynamic portfolio strategies by computing the upper and

lower bounds on the original expected utility function. In general, a better suboptimal solution implies a narrower gap from optimality. In this case, the problem can be formulated as a dual problem to solve the original problem by the convex duality [4][9].

Here in the following sections, we would like to consider the case of exponential utility using the Martingale approach. We start by defining the market below.

2.2.2 Probability Techniques

Definition 2.2.2.1. *A discrete supermartingale is a sequence X_1, X_2, X_3, \dots of integrable random variables satisfying*

$$\mathbb{E}[X_{n+1}|X_1, \dots, X_n] \leq X_n.$$

Definition 2.2.2.1. *Martingale Optimality Principle: If there exists a control strategy $u^*(\cdot)$ such that the utility function*

$$f(t, x) := \mathbb{E}_{t,x} \left[F \left(X^{u^*}(T) \right) \right] \quad (2.11)$$

satisfying:

1. $f(t, X^{u^*}(t))$ is a martingale;
2. $f(t, X^u(t))$ is a supermartingale for any admissible strategy $u(\cdot)$.

then we can obtain that $u^(\cdot)$ is an optimal strategy for maximising the utility (2.11).*

The proof of Martingale Optimality Principle can be found in [12].

2.2.3 The Market Model

Consider that the market consists of a riskless asset which is normalised to be $B_t = 1$, i.e. $r = 0$. We now consider risky asset as a continuous semimartingale. The stock price process is assumed to follow the SDE:

$$dS_t = \mu_t S_t dt + \sigma_t S_t dW_t$$

where μ and σ satisfy

$$\int_0^T (|\mu_t| + |\sigma_t^2|) dt < \infty$$

to ensure the integrals in the dynamic of stock are well defined.

Denote π_t^0 as the amount invested in the riskless asset at time t , π_t^1 as the amount invested in the risky asset at time t . We put $c_t = 0$ which means no consumption costs of wealth in the market. Under the self-financing condition, a portfolio's changing is only followed by gains and losses in the trading. Thus, we can define the wealth process here as

$$X_t^\pi = x + \int_0^t \pi_s dS_s$$

with initial condition $x := \pi_0^0 + \pi_0^1 S_0$.

2.2.4 Utility Function

We represent an investor's preference over the terminal wealth as a utility function $U : \mathbb{R} \rightarrow \mathbb{R}$ which is smooth, strictly concave and continuously differentiable and satisfying

$$U'(-\infty) = \infty, \quad U'(\infty) = 0.$$

Here a strictly increasing utility function implies a larger utility for more favourable payoffs. Also, by Jensen's inequality [8], concavity of the utility function implies

$\mathbb{E}[U(X)] \leq U[\mathbb{E}(X)]$ which means that the investor is risk-averse.

Also, the utility function is assumed to satisfy reasonable asymptotic elasticity, i.e.

$$\limsup_{x \rightarrow \infty} \frac{xU'(x)}{U(x)} < 1 \quad \text{and} \quad \liminf_{x \rightarrow -\infty} \frac{xU'(x)}{U(x)} > 1.$$

2.2.5 Optimal Portfolio Problem

We consider the terminal wealth process, the optimal portfolio problem is then denoted as

$$\text{maximizing} \quad \mathbb{E}[U(X_T^\pi)] \tag{2.12}$$

where π runs through the set of admissible trading strategies. It is not straightforward to obtain the optimiser of this problem. In the following section, we find it via a martingale method.

2.2.6 The Martingale Approach

For a fixed the time horizon $T \in [0, \infty]$, S_t is a stochastic process on a probability space $(\Omega, \mathbb{F}, \mathbb{P})$ modelling the stock price in the market. As mentioned in previous sections, $S = S_t$ denotes a locally bounded semimartingale. A probability measure equivalent to the original measure $\mathbb{Q} \ll \mathbb{P}$ is an equivalent local martingale measure for S such that S is a local martingale in measure \mathbb{Q} . We denoted $\mathbb{M}^\alpha(S)$ as the family of absolutely continuous local martingale measure and it follows that

$$\mathbb{M}^\alpha(S) \neq \emptyset.$$

For a utility function defined above, we define the convex dual $V(y)$ be

$$V(y) := \sup_x \left(U(x) - xy \right),$$

the Legendre transform of $-U(-x)$, where x is in the domain of utility function U and y is nonnegative. Thus, we have the relation $U' = (-V')^{-1}$. By the definition of $V(y)$,

$$U(x) \leq V(y) + xy$$

holds for any x runs through the domain of U and nonnegative y . Hence, for any $y > 0$ and for any equivalent martingale measure \mathbb{Q} , we have the duality upper bound

$$\begin{aligned} \mathbb{E}[U(X_T^\pi)] &\leq \mathbb{E} \left[V \left(y \frac{d\mathbb{Q}(y)}{d\mathbb{P}} \right) \right] + y \mathbb{E}_{\mathbb{Q}(y)}[X_T] \\ &\leq \mathbb{E} \left[V \left(y \frac{d\mathbb{Q}(y)}{d\mathbb{P}} \right) \right] + yx \end{aligned}$$

holds for any strategy π that its wealth process X^π is a supermartingale under the equivalent martingale measure \mathbb{Q} . We assume that such equivalent martingale measure always exists. Suppose there exists a strategy $\hat{\pi}_t$ and a martingale measure $\hat{\mathbb{Q}}$ satisfying

$$\hat{y} \frac{d\hat{\mathbb{Q}}}{d\mathbb{P}} = U'(X_T^{\hat{\pi}}) \quad \text{and} \quad \mathbb{E}_{\hat{\mathbb{Q}}}[\hat{X}_T^{\hat{\pi}}] = x. \quad (2.13)$$

Then $\hat{\pi}$ maximises the expected utility $\mathbb{E}[U(X_T^\pi)]$ over any possible sets of wealth processes that are supermartingale under measure $\hat{\mathbb{Q}}$.

Then, the primal optimisation problem (2.12) can turn into problem of finding a solution of

$$v(y) = \inf_{\mathbb{Q} \in \mathbb{M}^\alpha(S)} \mathbb{E} \left[V \left(y \frac{d\mathbb{Q}}{d\mathbb{P}} \right) \right].$$

Consider the original optimisation problem

$$u(x) = \sup \mathbb{E} \left[U \left(x + \int_0^T \pi_t dS_t \right) \right],$$

for all admissible π_t , the optimal solution $\hat{\pi}$ exists and is unique. Denote $y = u'(x)$, there exists a positive \hat{y} such that the dual problem $\mathbb{E} \left[V \left(\hat{y} \frac{dQ}{dP} \right) \right]$ is minimised by a so-called dual minimiser \hat{Q} . Relation (2.13) can be described as

$$x + \int_0^T \hat{\pi}_t dS_t = -V' \left(\hat{y} \frac{d\hat{Q}}{dP} \right) \quad \text{and} \quad \hat{y} \frac{d\hat{Q}}{dP} = U' \left(x + \int_0^T \hat{\pi}_t dS_t \right). \quad (2.14)$$

2.2.7 Special Case of Exponential Utility

Here, we consider the exponential case with

$$U(x) = -e^{-\lambda x}$$

for positive λ . We can define the conjugate function as

$$V(y) = \frac{y}{\lambda} \left(\log \left(\frac{y}{\lambda} \right) - 1 \right),$$

relation (3) can be specialised to

$$X_T^{\hat{\pi}} = -\frac{1}{\lambda} \log \left(\frac{y}{\lambda} \frac{d\hat{Q}}{dP} \right) \quad \text{and} \quad \frac{d\hat{Q}}{dP} = \frac{\lambda}{y} e^{-\lambda X_T^{\hat{\pi}}}.$$

Under a Black Scholes Model, the equivalent martingale measure is given by the density

$$\frac{dQ}{dP} = \exp \left(- \int_0^T \frac{\mu}{\sigma} dW_t - \int_0^T \frac{\mu^2}{2\sigma^2} dt \right),$$

where $\frac{\mu}{\sigma}$ is the Sharpe ratio with interest rate $r = 0$. The minimal \hat{y} can be determined as

$$\begin{aligned}\hat{y} &= \lambda \exp(-\lambda x - \mathbb{E}[\frac{d\hat{\mathbb{Q}}}{d\mathbb{P}} \log(\frac{d\hat{\mathbb{Q}}}{d\mathbb{P}})]) \\ &= \lambda \exp(-\lambda x - \mathbb{E}_{\hat{\mathbb{Q}}}[-\int_0^T \frac{\mu}{\sigma} dW_t - \int_0^T \frac{\mu^2}{2\sigma^2} dt]) \\ &= \lambda \exp(-\lambda x - \frac{\mu^2}{2\sigma^2} T)\end{aligned}$$

and the duality bound is given by

$$\begin{aligned}\mathbb{E}[-e^{-\lambda X_T^\pi}] &\leq \exp(-\lambda x - \mathbb{E}[\frac{d\hat{\mathbb{Q}}}{d\mathbb{P}} \log(\frac{d\hat{\mathbb{Q}}}{d\mathbb{P}})]) \\ &= -\exp(-\lambda x - \frac{\mu^2}{2\sigma^2} T).\end{aligned}\tag{2.15}$$

According to condition (2.14), the optimal strategy $\hat{\pi}$ can be deduced from

$$\begin{aligned}X_T^{\hat{\pi}} &= x + \int_0^T \pi_t dS_t \\ &= \frac{1}{\lambda} \log(\frac{\hat{y}}{\lambda} \frac{d\mathbb{Q}}{d\mathbb{P}}) \\ &= -\frac{\log \hat{y}/\lambda}{\lambda} + \int_0^T \frac{\mu}{\lambda\sigma} dW_t + \int_0^T \frac{\mu^2}{\lambda 2\sigma^2} dt \\ &= x + \int_0^T \frac{\mu}{\lambda\sigma^2} \frac{1}{S_t} (S_t \sigma dW_t + S_t \mu dt).\end{aligned}$$

Thus, the optimal strategy satisfied

$$\hat{\pi}_t = \frac{\mu}{\lambda\sigma^2} \frac{1}{S_t}.\tag{2.16}$$

2.2.8 PDE Approach to Exponential Utility Problem

Here, we would like to solve the exponential utility problem via stochastic control approach. Starting from initial value x , the value process can be described as

$$J(t, x) = \operatorname{ess\,sup}_{\pi} \mathbb{E} \left[U \left(x + \int_t^T \pi_u dS_u \right) \middle| \mathcal{F}_t \right]$$

which seeks the maximal utility over all π_t in the time horizon $[t, T]$. Consider any possible trading strategy π , assume the above supremum is obtained by some strategy $\hat{\pi}$. For $0 \leq s \leq t \leq T$, it follows that

$$\begin{aligned} \mathbb{E}[J(t, X_t^\pi) | \mathcal{F}_s] &= \mathbb{E}[\mathbb{E}[U(X_t^\pi + \int_t^T \hat{\pi}_u dS_u) | \mathcal{F}_t] | \mathcal{F}_s] \\ &= \mathbb{E}[\mathbb{E}[U(X_s^\pi + \int_s^t \pi_u dS_u + \int_t^T \hat{\pi}_u dS_u) | \mathcal{F}_t] | \mathcal{F}_s] \\ &\leq \mathbb{E}[\mathbb{E}[U(X_s^\pi + \int_s^t \hat{\pi}_u dS_u + \int_t^T \hat{\pi}_u dS_u) | \mathcal{F}_t] | \mathcal{F}_s] \\ &= J(s, X_s^\pi), \end{aligned}$$

which shows that the value process $J(t, X_t^\pi)$ is a martingale and is a supermartingale under $\hat{\pi}$. Hence, by the martingale optimality principle (lemma 2.2.2.1.), the equality holds for the optimal trading strategy $\hat{\pi}$. By the Itô formula, we can differentiate the value function $J(t, X_t^\pi)$ as

$$dJ(t, X_t^\pi) = (J_t + \mu J_x \pi_t S_t + \frac{1}{2} \sigma^2 J_{xx} (\pi_t S_t)^2) dt + \sigma J_x \pi_t S_t dW_t.$$

To obtain the optimiser, we notice that the drift rate is a quadratic function in risky asset $\pi_t S_t$. This yields the maximised risky position

$$\hat{\pi}_t S_t = -\frac{\mu J_x}{\sigma^2 J_{xx}}. \quad (2.17)$$

By the martingale optimality principle, the drift term should vanish for the optimiser. Substituting the equation (2.17) back into the drift term, this leads to the following partial differential equation of J

$$J_t = \frac{\mu^2 J_x^2}{2\sigma^2 J_{xx}}.$$

In particular, in the case of exponential utility, the value process can be calculated as

$$J(t, x) = \operatorname{ess\,sup}_{\pi} \mathbb{E}[-e^{-\lambda(x + \int_t^T \pi_u dS_u)} | \mathcal{F}_t] = e^{-\lambda x} j(t)$$

where $j(t) = J(t, 0)$ is independent of initial value x . Then the PDE can be reduced into the following ODE

$$j' = \frac{\mu^2}{2\sigma^2} j$$

which yields the solution of

$$j(t) = -\exp\left(\frac{\mu^2}{2\sigma^2}(T - t)\right)$$

where the terminal condition $j(T) = -1$. Thus, the maximal utility satisfies

$$J(0, x) = -\exp\left(-\lambda x - \frac{\mu^2}{2\sigma^2}T\right)$$

as in the previous section. In particular, the optimal strategy can be obtained by

$$\hat{\pi}_t = \frac{\mu}{\sigma^2 \lambda} \cdot \frac{1}{S_t}. \tag{2.18}$$

In this section, we solve the exponential utility problem via stochastic control approach. The optimal strategy in equation (2.18) yields the same result as the martingale approach in equation (2.18). In summary, in the case of exponential utility, closed form solutions for the optimal investment strategy as well as the maximum

utility function are obtained.

3 Mean-Variance Asset-Liability Portfolio Selection

In portfolio selection problems, Markowitz model and Merton's problem focus only on the asset sides. However, these studies fail to consider the liabilities. Recently, asset allocation involving liabilities becomes an essential part of the Asset Liability Management. Sharpe and Tint (1990) [21] first studied the Asset-Liability management in a mean variance criteria under a static model. In [11], Keel and Müller (1995) showed that liabilities bring a parallel shift of the efficient frontier. Later on, Leippold, Trojani and Vanini (2004) [13] presented a geometric approach to the multi-period setting using embedding technique from Li and Ng (1998) [14]. Chiu and Li (2006) [6] further developed the ALM under mean-variance approach into a continuous case by adopting the framework of stochastic linear quadratic control, where the dynamics of assets and liabilities follows a geometric Brownian Motions. Xie and Li (2007) [22], Xie, Li and Wang (2008) [23] extended the liability process to those governed by a Brownian Motion with drifts. Chen, Yang and Yin (2008) [5] investigated a continuous model with regime switching by duality theory.

In this chapter, we shall study the Mean-Variance Asset-Liability portfolio selection problem in discrete framework. We start with a single step optimisation problem, and then develop it into a multi step problem in section 3.2.

3.1 Mean Variance Asset Liability Management – One Step Problem

3.1.1 Market Model

For any asset i , we denote ψ_t as the value at time t . After Δt period of time, we assume the price of asset follows

$$\begin{aligned}\psi_{t+\Delta t} &= \psi_t + \Delta\psi_t \\ &= \psi_t + (\mu_t\psi_t + c_t)\Delta t + (\sigma_t\psi_t + d_t)\Delta W_t,\end{aligned}$$

where μ_t and σ_t are the appreciation-rate and volatility of the process; c_t and d_t are given parameters of asset i at time t ; Δt denotes the time interval and ΔW_t denotes the randomness.

Let $\boldsymbol{\psi}_t = (\psi_{t,1}, \psi_{t,2}, \dots, \psi_{t,n_A}, \dots, \psi_{t,n_A+n_L})^T$ denote all financial assets and liabilities in the financial market. Suppose there are n_A tradable assets, these assets are labelled as $\psi_{t,1}, \psi_{t,2}, \dots, \psi_{t,n_A}$. Suppose there are n_L liabilities in the financial market, these liabilities are labelled as $\psi_{t,n_A+1}, \psi_{t,n_A+2}, \dots, \psi_{t,n_A+n_L}$.

Hence we have the vector $\boldsymbol{\psi}_{t+\Delta t} = (\psi_{t+\Delta t,1}, \psi_{t+\Delta t,2}, \dots, \psi_{t+\Delta t,n_A}, \dots, \psi_{t+\Delta t,n_A+n_L})^T$ follows:

$$\boldsymbol{\psi}_{t+\Delta t} = (\mathbf{I} + \boldsymbol{\mu}\Delta t + \boldsymbol{\sigma}\Delta\mathbf{W}_t)\boldsymbol{\psi}_t + \mathbf{c}\Delta t + \Delta\mathbf{W}_t\mathbf{d} \quad (3.1)$$

where \mathbf{I} is an identity matrix with ones on the diagonal and zeros elsewhere; $\boldsymbol{\mu}$ is a diagonal matrix with $\mu_{t,i}$ on the diagonal and $\boldsymbol{\sigma}$ is an $(n_A + n_L) \times (n_A + n_L)$ matrix denoting the volatility; \mathbf{c} is a vector of $c_{t,i}$ and \mathbf{d} is a vector with components $d_{t,i}$; $\Delta\mathbf{W}_t$ is a diagonal matrix with $\Delta W_{t,i}$ on the diagonal, denoting the randomness.

We further denote $\mathbf{u}_t = (u_{t,1}, u_{t,2}, \dots, u_{t,n_A}, -1, \dots, -1)$ be the chosen shares of holding asset i . For liabilities $\psi_{t,n_A+1}, \psi_{t,n_A+2}, \dots, \psi_{t,n_A+n_L}$, we always assume that $u_{t,l} = -1$, for $l = n_A + 1, n_A + 2, \dots, n_A + n_L$.

Then we can have the total wealth of holding at time $t + \Delta t$ as

$$\begin{aligned} X_{t+\Delta t} &= \boldsymbol{\psi}_{t+\Delta t}^T \mathbf{u}_t \\ &= \sum_i [(1 + \mu_{t,i}\Delta t + \sigma_{t,i}\Delta W_{t,i})\psi_{t,i} + c_{t,i}\Delta t + d_{t,i}\Delta W_{t,i}]u_{t,i} \end{aligned}$$

with its mean and variance:

$$\mathbb{E}[X_{t+\Delta t}] = \sum_i [(1 + \mu_{t,i}\Delta t)\psi_{t,i} + c_{t,i}\Delta t]u_{t,i} \quad (3.2)$$

$$Var[X_{t+\Delta t}] = \sum_i \sum_j [(\sigma_{t,i}\psi_{t,i} + d_{t,i})u_{t,i}] [(\sigma_{t,j}\psi_{t,j} + d_{t,j})u_{t,j}] \rho_{ij}\Delta t. \quad (3.3)$$

where ρ_{ij} denotes the correlation of two Brownian motions, i.e. $\Delta W_{t,i}\Delta W_{t,j} = \rho_{ij}\Delta t$.

3.1.2 Mean Variance Optimization – One Step, Multi Asset Problem

From equation (3.2) and equation (3.3), we can write our mean variance utility function as:

$$\begin{aligned} J &= \mathbb{E}[X_{t+\Delta t}] - \frac{\gamma}{2} Var[X_{t+\Delta t}] \\ &= \sum_i [(1 + \mu_{t,i}\Delta t)\psi_{t,i} + c_{t,i}\Delta t]u_{t,i} \\ &\quad - \frac{\gamma}{2} \sum_i \sum_j [(\sigma_{t,i}\psi_{t,i} + d_{t,i})u_{t,i}] [(\sigma_{t,j}\psi_{t,j} + d_{t,j})u_{t,j}] \rho_{ij}\Delta t. \end{aligned} \quad (3.4)$$

Define coefficient $\xi_{t,i}$ and $\Omega_{t,i}$ in terms of market parameters μ, σ, a, b and the current asset price ψ as:

$$\xi_{t,i} = (1 + \mu_{t,i}\Delta t)\psi_{t,i} + a_{t,i}\Delta t \quad (3.5)$$

$$\Omega_{t,ij} = (\sigma_{t,i}\psi_{t,i} + b_{t,i})(\sigma_{t,j}\psi_{t,j} + b_{t,j})M_{ij}. \quad (3.6)$$

Let vector $\boldsymbol{\xi}_t = (\xi_{t,1}, \xi_{t,2}, \dots, \xi_{t,n_A}, \dots, \xi_{t,n_L+n_A})^T$ with components $\xi_{t,i}$ as in equation (3.5) denotes the expected growth on each asset. Let matrix $\boldsymbol{\Omega}_t$ with components $\Omega_{t,ij}$ as in equation (3.6) denotes the randomness and correlation among assets. Then mean variance utility function in equation (3.4) can be written in vector form:

$$J(t, X_t, u_t) = \boldsymbol{\xi}_t^T \mathbf{u}_t - \frac{\gamma}{2} \mathbf{u}_t^T \boldsymbol{\Omega}_t \mathbf{u}_t. \quad (3.7)$$

We put further constraints on the optimisation problem. Assuming at time t , we are holding X_t amount of wealth that can be allocated to various assets. For each liability l , assume that we always hold -1 share of it. Under these settings, our mean-variance optimization problem becomes:

$$\text{maximizing} \quad \boldsymbol{\xi}_t^T \mathbf{u}_t - \frac{\gamma}{2} \mathbf{u}_t^T \boldsymbol{\Omega}_t \mathbf{u}_t \quad (3.8)$$

$$\text{s.t.} \quad \boldsymbol{\psi}_t^T \mathbf{u}_t = X_t, \quad (3.9)$$

$$u_{t,l} = -1, \quad \text{for liabilities } l = n_A + 1, n_A + 2, \dots, n_A + n_L \quad (3.10)$$

where X_t is the initial total wealth at time t .

Lemma 3.1.2.1. Optimal Solution

Define $\bar{\boldsymbol{\psi}}_t$ as an $(n_A \times 1)$ vector with components $\bar{\psi}_{t,i} = \psi_{t,i}$ denotes the risky assets prices; $\bar{\mathbf{u}}_t$ as an $(n_A \times 1)$ vector with components $\bar{u}_{t,i} = u_{t,i}$ denotes the holding shares of risky assets; \bar{X}_t as the wealth of all risky assets with $\bar{X}_t = X_t - \sum_{i=n_A+1}^{n_A+n_L} \psi_i u_i = \bar{\boldsymbol{\psi}}_t^T \bar{\mathbf{u}}_t$; $\bar{\boldsymbol{\Omega}}_t$ as an $(n_A \times n_A)$ matrix with components $\bar{\Omega}_{t,i} = \Omega_{t,i}$; $\bar{\boldsymbol{\xi}}_t$ as an $(n_A \times 1)$ vector with components $\bar{\xi}_{t,i} = \xi_{t,i}$. Furthermore, we write $\bar{\mathbf{P}}_t = \bar{\boldsymbol{\xi}}_t + \gamma \bar{\bar{\boldsymbol{\Omega}}}_t$ where $\bar{\bar{\boldsymbol{\Omega}}}_{t,i} = \sum_{j=n_A+1}^{n_A+n_L} \Omega_{t,ij}$ is an $(n_A \times 1)$ vector.

The mean-variance optimization problem J subjected to constraint (3.9) and con-

straint (3.10) can be optimised by choosing

$$\begin{cases} \bar{\mathbf{u}}_t^* = \mathbf{C}_t \bar{X}_t + \mathbf{h}_t, \\ u_{t,i}^* = -1, \end{cases} \quad i = n_A + 1, n_A + 2, \dots, n_L + n_A. \quad (3.11)$$

Coefficients $C_{t,i}$ and $h_{t,i}$ are independent of the current wealth X_t and can be obtained by

$$\mathbf{C}_t = \frac{\bar{\Omega}_t^{-1} \bar{\psi}_t}{\bar{\psi}_t^T \bar{\Omega}_t^{-1} \bar{\psi}_t} \quad (3.12)$$

$$\mathbf{h}_t = \frac{1}{\gamma} \left[\bar{\Omega}_t^{-1} \bar{P}_t - \frac{\bar{\Omega}_t^{-1} \bar{\psi}_t \cdot \bar{\psi}_t^T \bar{\Omega}_t^{-1} \bar{P}_t}{\bar{\psi}_t^T \bar{\Omega}_t^{-1} \bar{\psi}_t} \right]. \quad (3.13)$$

Proof of Lemma 3.1.2.1.

In order to simplify our mean-variance optimisation problem (3.8), we substitute constraint (3.10) into the utility function (3.4) by replacing $u_{t,i}$ with -1 for $i = n_A + 1, n_A + 2, \dots, n_L + n_A$. Then the optimisation problem can be reformulated as

$$\text{maximizing} \quad \boldsymbol{\xi}_t^T \mathbf{u}_t - \frac{\gamma}{2} \mathbf{u}_t^T \boldsymbol{\Omega}_t \mathbf{u}_t \quad (3.14)$$

$$\text{s.t.} \quad \bar{X}_t = \bar{\psi}_t^T \bar{\mathbf{u}}_t. \quad (3.15)$$

Now, we would like to find the optimal solution for this problem by the Lagrangian Method. Define the Lagrangian function as:

$$\mathcal{L} = \boldsymbol{\xi}_t^T \mathbf{u}_t - \frac{\gamma}{2} \mathbf{u}_t^T \boldsymbol{\Omega}_t \mathbf{u}_t + \lambda (\bar{X}_t - \bar{\psi}_t^T \bar{\mathbf{u}}_t) \quad (3.16)$$

where λ is the Lagrange multiplier for constraint (3.15).

For the first-order conditions, we differentiate \mathcal{L} with respect to $u_{t,i}$ and λ respectively

and set the derivatives to zero.

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial u_{t,i}} = 0 &\implies \xi_{t,i} - \gamma \sum_{j=1}^{n_A+n_L} \Omega_{t,ij} u_j - \lambda \psi_{t,i} = 0 \\ &\implies \bar{\xi}_t - \gamma \bar{\Omega}_t \bar{\mathbf{u}}_t + \gamma \bar{\bar{\Omega}}_t - \lambda \bar{\psi}_t = 0 \end{aligned} \quad (3.17)$$

$$\implies \bar{\mathbf{u}}_t = \frac{1}{\gamma} \bar{\Omega}_t^{-1} (\bar{\xi}_t + \gamma \bar{\bar{\Omega}}_t - \lambda \bar{\psi}_t) \quad (3.18)$$

$$\text{for invertible matrix } \bar{\Omega}. \quad (3.19)$$

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \lambda} = 0 &\implies \bar{X}_t - \bar{\psi}_t^T \bar{\mathbf{u}}_t = 0 \\ &\implies \bar{X}_t = \bar{\psi}_t^T \bar{\mathbf{u}}_t. \end{aligned} \quad (3.20)$$

To determine λ , we insert equation (3.18) to equation (3.20), we then can have

$$\lambda = \frac{1}{\bar{\psi}_t^T \bar{\Omega}_t^{-1} \bar{\psi}_t} (\bar{\psi}_t^T \bar{\Omega}_t^{-1} \bar{\mathbf{P}}_t - \gamma \bar{X}_t). \quad (3.21)$$

Insert λ from equation (3.21) back to equation (3.18), we can compute the optimal control $\bar{\mathbf{u}}_t^*$ as

$$\bar{\mathbf{u}}_t^* = \mathbf{C}_t \bar{X}_t + \mathbf{h}_t, \quad (3.22)$$

where \mathbf{C}_t and \mathbf{h}_t can be obtained by

$$\mathbf{C}_{t,i} = \frac{\bar{\Omega}_t^{-1} \bar{\psi}_t}{\bar{\psi}_t^T \bar{\Omega}_t^{-1} \bar{\psi}_t} \quad (3.23)$$

$$\mathbf{h}_{t,i} = \frac{1}{\gamma} \left[\bar{\Omega}_t^{-1} \bar{\mathbf{P}}_t - \frac{\bar{\Omega}_t^{-1} \bar{\psi}_t \cdot \bar{\psi}_t^T \bar{\Omega}_t^{-1} \bar{\mathbf{P}}_t}{\bar{\psi}_t^T \bar{\Omega}_t^{-1} \bar{\psi}_t} \right]. \quad (3.24)$$

Corollary 3.1.2.1. *According to Lemma 3.1.2.1, the maximum expected utility is*

$$\begin{aligned}
J^{u*}(t, \bar{X}_t) = & -\frac{\gamma}{2\bar{\psi}_t^T \bar{\Omega}_t^{-1} \bar{\psi}_t} \bar{X}_t^2 \\
& + \frac{1}{\bar{\psi}_t^T \bar{\Omega}_t^{-1} \bar{\psi}_t} \left[\bar{\xi}_t^T \bar{\Omega}_t^{-1} \bar{\psi}_t + \frac{1}{2} (\bar{\psi}_t^T \bar{\Omega}_t^{-1} \bar{P}_t - \bar{P}_t^T \bar{\Omega}_t^{-1} \bar{\psi}_t) + \gamma \bar{\psi}_t^T (\bar{\Omega}_t^{-1})^T \bar{\bar{\Omega}}_t \right] \bar{X}_t \\
& + \frac{1}{\gamma} \left[\bar{\xi}_t^T \bar{\Omega}_t^{-1} \bar{P}_t - \frac{\bar{\psi}_t^T \bar{\Omega}_t^{-1} \bar{P}_t}{\bar{\psi}_t^T \bar{\Omega}_t^{-1} \bar{\psi}_t} \bar{\xi}_t^T \bar{\Omega}_t^{-1} \bar{\psi}_t \right] - \bar{\xi}_t - \frac{\gamma}{2} \bar{\bar{\Omega}}_t \\
& + \frac{1}{2\gamma} \left[\bar{P}_t^T \bar{\Omega}_t^{-1} \bar{P}_t - \frac{\bar{\psi}_t^T \bar{\Omega}_t^{-1} \bar{P}_t}{\bar{\psi}_t^T \bar{\Omega}_t^{-1} \bar{\psi}_t} \bar{P}_t^T \bar{\Omega}_t^{-1} \bar{\psi}_t \right] \\
& + \left[\bar{P}_t^T (\bar{\Omega}_t^{-1})^T \bar{\bar{\Omega}}_t - \frac{\bar{\psi}_t^T \bar{\Omega}_t^{-1} \bar{P}_t}{\bar{\psi}_t^T \bar{\Omega}_t^{-1} \bar{\psi}_t} \bar{\psi}_t^T (\bar{\Omega}_t^{-1})^T \bar{\bar{\Omega}}_t \right]
\end{aligned} \tag{3.25}$$

where $\bar{\xi}_t = \sum_{i=n_A+1}^{n_A+n_L} \xi_{t,i}$ and $\bar{\bar{\Omega}}_t = \sum_{i=n_A+1}^{n_A+n_L} \sum_{j=n_A+1}^{n_A+n_L} \Omega_{t,ij}$. The function gives a quadratic utility of the form:

$$J^{u*}(t, \bar{X}_t) = p_{t,1} \bar{X}_t^2 + p_{t,2} \bar{X}_t + p_{t,3}$$

which is quadratic in current wealth X_t and the coefficients $p_{t,1}, p_{t,2}, p_{t,3}$ dependent only on the current status t .

Definition 3.1.2.1. *The Principle of Optimality [2] is presented as follows:*

An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision.

The Principle of Optimality is also known as the Bellman Principle.

From Corollary 3.1.2.1, in this very general model of assets, the maximum utility is nonlinear in the current wealth X_t and the current asset prices ψ_t . If we are going to deduce the model for a multi-step problem backwardly, we may fail to do so due to the nonlinearity. In this case, the Bellman Principle as describe in Definition 3.1.2.1 may not work. Furthermore, the optimal solution relies on the invertibility of the matrix $\bar{\Omega}$. This also make it complicated to look for an explicit solution. In the

following sections, we are looking for some special cases that the utility may result in a linearity in the wealth and the asset prices.

Theorem 3.1.2.1. *If the following conditions hold, the maximum utility in Corollary 3.1.2.1 becomes quadratic in current wealth X_t and current asset prices ψ_t :*

- a) *The growth of asset price is linear with the current price, i.e. vector \mathbf{c} and \mathbf{d} in equation (3.1) become 0;*
- b) *The market has only one risky asset.*

3.1.3 Special Case Study: One riskless asset and one risky asset

We consider a special case with no liabilities and only one riskless asset and one risky asset. We denote $n_L = 0$ which means no liabilities are involved in the market. We denote ψ_1 to be the riskless asset with drift $\mu_{t,1} \neq 0$, $c_{t,1} = 0$ and $\sigma_{t,1} = 0, d_{t,1} = 0$, corresponding to receive fix interests with no randomness. We denote ψ_2 as the risky asset follows a geometry Brownian motion with $\mu_{t,2}$ and $\sigma_{t,2}$ representing the drift and volatility. Hence, the vector of $\psi_{t+\Delta t} = [\psi_{t+\Delta t,1}, \psi_{t+\Delta t,2}]$ can be written as

$$\begin{pmatrix} \psi_{t+\Delta t,1} \\ \psi_{t+\Delta t,2} \end{pmatrix} = \begin{pmatrix} 1 + \mu_{t,1}\Delta t & 0 \\ 0 & 1 + \mu_{t,2}\Delta t \end{pmatrix} \begin{pmatrix} \psi_{t,1} \\ \psi_{t,2} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \sigma_{t,2}\Delta W \end{pmatrix} \begin{pmatrix} \psi_{t,1} \\ \psi_{t,2} \end{pmatrix}.$$

To obtained a maximum utility as in equation (3.8), we can choose the optimum strategy as

$$u_{t,1}^* = \frac{1}{\psi_{t,1}} [X_t - \frac{\mu_{t,2} - \mu_{t,1}}{\gamma \sigma_{t,2}^2}] \quad (3.26)$$

$$u_{t,2}^* = \frac{\mu_{t,2} - \mu_{t,1}}{\gamma \sigma_{t,2}^2 \psi_{t,2}} \quad (3.27)$$

Then the maximum expected utility is

$$J^{u^*}(t, X_t) = (1 + \mu_{t,1}\Delta t)X_t + \frac{(\mu_{t,2} - \mu_{t,1})^2\Delta t}{2\gamma\sigma_{t,2}^2}. \quad (3.28)$$

Maximum utility only depends on the current total wealth X_t and the market parameters.

3.1.4 Special Case Study: One riskless asset, one risky asset and multi Liabilities

If we consider the case that liabilities are involved and the market has only one riskless asset and one risky asset. Denote ψ_1 as the riskless asset with drift $\mu_{t,1} \neq 0$, $c_{t,1} = d_{t,1} = 0$ and $\sigma_{t,1} = 0$, corresponding to receive fix interest payments with no randomness. Let ψ_{Li} be the Liabilites follows a geometry Brownian motion with $\mu_{t,Li}$ and $\sigma_{t,Li}$ representing the drift and volatility, for $i = 1, 2, \dots, n_L$ up to n_L liabilities. Let ψ_2 be the risky asset follows a geometry Brownian motion with $\mu_{t,2}$ and $\sigma_{t,2}$ representing the drift and volatility. Then vector $\psi_{t+\Delta t} = [\psi_{t+\Delta t,1}, \psi_{t+\Delta t,2}, \psi_{t+\Delta t,L1}, \psi_{t+\Delta t,L2}, \dots, \psi_{t+\Delta t,Ln_L}]$ can be written as

$$\begin{aligned} \psi_{t+\Delta t} = & \begin{pmatrix} 1 + \mu_{t,1}\Delta t & 0 & \cdots & 0 \\ 0 & 1 + \mu_{t,2}\Delta t & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 + \mu_{t,Ln_L}\Delta t \end{pmatrix} \begin{pmatrix} \psi_{t,1} \\ \psi_{t,2} \\ \vdots \\ \psi_{t,Ln_L} \end{pmatrix} \\ & + \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & \sigma_{t,2}\Delta W & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_{t,Ln_L}\Delta W \end{pmatrix} \begin{pmatrix} \psi_{t,1} \\ \psi_{t,2} \\ \vdots \\ \psi_{t,Ln_L} \end{pmatrix}. \end{aligned} \quad (3.29)$$

To obtained a maximum utility as in equation (3.8), we can choose the optimum

strategy as

$$u_{t,1}^* = \frac{1}{\psi_{t,1}} \left[X_t - \frac{\mu_{t,2} - \mu_{t,1}}{\gamma \sigma_{t,2}^2} + (\mathbf{1}_{t,L} - \boldsymbol{\theta}_{t,L})^T \boldsymbol{\psi}_{t,L} \right] \quad (3.30)$$

$$u_{t,2}^* = \frac{1}{\psi_{t,2}} \left[\frac{\mu_{t,2} - \mu_{t,1}}{\gamma \sigma_{t,2}^2} + \boldsymbol{\theta}_{t,L}^T \boldsymbol{\psi}_{t,L} \right] \quad (3.31)$$

where $\boldsymbol{\theta}_{t,L} = [\frac{\sigma_{t,L1}}{\sigma_{t,2}} \rho_{2L1}, \frac{\sigma_{t,L1}}{\sigma_{t,2}} \rho_{2L2}, \dots, \frac{\sigma_{t,L1}}{\sigma_{t,2}} \rho_{2n_L}]$ is an n_L -vector denotes the correlation between the risky asset and liabilities; $\boldsymbol{\psi}_{t,L} = [\psi_{t,L1}, \psi_{t,L2}, \dots, \psi_{t,n_L}]$ is an n_L -vector denotes the current price of liabilities; $\mathbf{1}_L$ is an n_L all ones vector. Then the utility function can be maximised as

$$\begin{aligned} J^{u^*}(t, X_t) = & (1 + \mu_{t,1} \Delta t) X_t + \frac{(\mu_{t,2} - \mu_{t,1})^2 \Delta t}{2\gamma \sigma_{t,2}^2} \\ & - \frac{\gamma}{2} \boldsymbol{\psi}_{t,L}^T \mathbf{K}_t \boldsymbol{\psi}_{t,L} + \mathbf{V}_t^T \boldsymbol{\psi}_{t,L} \end{aligned} \quad (3.32)$$

where \mathbf{K}_t is an $n_L \times n_L$ matrix with components $K_{t,ij} = \sigma_{t,Li} \sigma_{t,Lj} (\rho_{LiLj} - \rho_{2Li} \rho_{2Lj}) \Delta t$; \mathbf{V}_t is an n_L vector with components $V_{t,i} = \frac{(\mu_{t,2} - r) \Delta t}{\sigma_{t,2}} \sigma_{t,Li} \rho_{2Li} + (r - \mu_{t,Li}) \Delta t$.

Assume there is one riskless asset, one risky asset and only one liability in the market. Then we can reduce the case into choosing $u_2^* = \frac{1}{\psi_{t,2}} \left(\frac{\mu_{t,2} - \mu_{t,1}}{\gamma \sigma_{t,2}^2} + \frac{\sigma_{t,L} \rho_{2L}}{\sigma_{t,2}} \psi_{t,L} \right)$ shares of risky asset and $u_1^* = \frac{1}{\psi_{t,1}} \left[X_t - \frac{\mu_{t,2} - \mu_{t,1}}{\gamma \sigma_{t,2}^2} + \left(1 - \frac{\sigma_{t,L} \rho_{2L}}{\sigma_{t,2}} \right) \psi_{t,L} \right]$ shares of riskless asset. Then the mean variance utility is optimised as

$$\begin{aligned} J(t, X_t)^{u^*} = & (1 + \mu_{t,1} \Delta t) X_t + \frac{(\mu_{t,2} - \mu_{t,1})^2 \Delta t}{2\gamma \sigma_{t,2}^2} \\ & - \frac{\gamma}{2} \sigma_{t,L}^2 \Delta t (1 - \rho_{2L}^2) \psi_{t,L}^2 + \left[\frac{(\mu_{t,2} - \mu_{t,1}) \Delta t}{\sigma_{t,2}} \sigma_{t,L} \rho_{2L} + (r - \mu_{t,L}) \Delta t \right] \psi_{t,L}, \end{aligned} \quad (3.33)$$

which is linear with current wealth X_t and quadratic in liability $\psi_{t,L}$.

3.2 Mean Variance Asset Liability Management – Multi Period Problem

In section (3.1.4), we have found the optimal solution for a single period problem with one riskless, one risky and multiple liability in the market. We learn from equation (3.32) that the maximum mean variance utility is linear in current wealth and quadratic in current liability prices. In this case, we would like to develop the results into a multi-period solution using backward induction.

To find out the optimal investment strategy with objective utility function $J(X_T) = \mathbb{E}[X_T] - \frac{\gamma}{2} \text{Var}[X_T]$, we solve backwardly. Indeed, we would find the sequence of optimal control u_t^* for $t = 0, 1, \dots, T - \Delta t$ backwardly in the following fashion:

1. $u_{T-\Delta t}^*$ is the optimal solution to the single period problem which maximise $J(X_T)$ for fixed $X_{T-\Delta t}, \psi_{T-\Delta t}$.
2. For fixed $u_{T-\Delta t}^*$, the maximum utility $J(X_T)$ is a function quadratic in $X_{T-\Delta t}, \psi_{T-\Delta t}$. If we fix $X_{T-2\Delta t}$ and $\psi_{T-2\Delta t}$, we shall find a $u_{T-2\Delta t}^*$ that maximises the objective $J(X_T)$. In this case, $u_{T-\Delta t}^*$ and $u_{T-2\Delta t}^*$ maximise $J(X_T)$ for fixed $u_{T-2\Delta t}^*$.
3. We repeat the above steps to find all u_t^* for $t = 0, 1, \dots, T - \Delta t$ under the backwardly.

3.2.1 Financial Market

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a fixed complete probability space. $\mathcal{F} := \cup_{t=0}^T \mathcal{F}_t$ is defined as a filtration for some finite T where T denotes the investment time horizon. Let \mathbb{E} denotes the expectation with respect to measure \mathbb{P} . We assume that no transaction costs or taxes are included in the financial market. Investors can trade among a riskless asset $\psi_{t,1}$ and one risky asset $\psi_{t,2}$ over time $t \in \tau := \{0, \Delta t, \dots, T\}$ and investors

are exposed to n_L liabilities $\psi_{t,L_1}, \psi_{t,L_2}, \dots, \psi_{t,L_{n_L}}$.

3.2.2 Financial Assets

Suppose there are 2 tradable assets in the financial market. Their prices are labelled as $\psi_{t,1}$ for a riskless asset and $\psi_{t,2}$ for a risky asset. Also, there are n_L liabilities in the market and their prices are labelled as $\psi_{t,L_1}, \psi_{t,L_2}, \dots, \psi_{t,L_{n_L}}$. The prices of these assets follow the same model as in equation (3.29): $t \in \tau := \{0, \Delta t, \dots, T - \Delta t, T\}$

$$\begin{aligned} \psi_{t+\Delta t} = & \begin{pmatrix} 1 + \mu_{t,1}\Delta t & 0 & \cdots & 0 \\ 0 & 1 + \mu_{t,2}\Delta t & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 + \mu_{t,L_{n_L}}\Delta t \end{pmatrix} \begin{pmatrix} \psi_{t,1} \\ \psi_{t,2} \\ \vdots \\ \psi_{t,L_{n_L}} \end{pmatrix} \\ & + \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & \sigma_{t,2}\Delta W_t & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_{t,L_{n_L}}\Delta W_t \end{pmatrix} \begin{pmatrix} \psi_{t,1} \\ \psi_{t,2} \\ \vdots \\ \psi_{t,L_{n_L}} \end{pmatrix}. \end{aligned} \quad (3.34)$$

To simplify our notations, we denote $r_{t,i} = 1 + \mu_{t,i}\Delta t$ as the growth of the i th asset and denote $\alpha_{t,i} = (\mu_{t,i} - \mu_{t,1})\Delta t$ and $D_{t,ij} = \sigma_{t,i}\sigma_{t,j}\rho_{ij}\Delta t$.

3.2.3 Wealth Process

Assume that investors can choose to invest their capital at both risky and riskless assets. Let u_t^i be the shares of investment in the i th asset at time t . Denote $u_t = (u_{t,2}, u_{t,2}, \dots, -1)^T$ as the vector of trading strategy through all assets. For liability L_i , we always assume holding -1 shares of it, i.e. $u_{t,L_i} = -1$. Let A_t^u denotes the total asset at time t under strategy u_t . Then the wealth of total assets of holding at

time t satisfies:

$$X_t = \boldsymbol{\psi}_t^T \mathbf{u}_t. \quad (3.35)$$

3.2.4 Mean Variance Criteria

The purpose of an investor is to choose a proper investment strategy in order to maximise the utility function under the mean-variance criteria. That is, the aim of an investor is to find \mathbf{u}_t such that the following objective functional of the mean-variance utility is of a maximum:

$$J(t, X_t, \psi_t; u) := \mathbb{E}_{t, X_t, \psi_t}[X_T^u] - \frac{\gamma}{2} \text{Var}_{t, X_t, \psi_t}[X_T^u], \quad (3.36)$$

where γ is a risk averse parameter; $\mathbb{E}_{t, X_t, \psi_t}[\cdot]$ and $\text{Var}_{t, X_t, \psi_t}[\cdot]$ denote the mean and variance of the terminal wealth X_T conditioned on the current wealth X_t and current liabilities ψ_t , i.e. $\mathbb{E}_{t, X_t, \psi_t}[X_T^u] = \mathbb{E}[X_N^u | X_t, \psi_t]$, $\text{Var}_{t, X_t, \psi_t}[X_T^u] = \text{Var}[X_N^u | X_t, \psi_t]$.

3.2.5 Optimal Solution

Define

$$g(t, X_t, \psi_t) := \mathbb{E}_{t, X_t, \psi_t}[X_T^{u^*}], \quad (3.37)$$

$$f(t, X_t, \psi_t) := \mathbb{E}_{t, X_t, \psi_t}[(X_T^{u^*})^2]. \quad (3.38)$$

Theorem 3.2.5.1. *The optimal control, u_t^* , for the objective function (3.36) subjected to the wealth process (3.35), can be determined as:*

$$u_{t,2}^* = \frac{1}{\psi_{t,2}} (C_t X_t + \boldsymbol{\eta}_t^T \boldsymbol{\psi}_{t,L} + h_t) \quad (3.39)$$

where coefficients $C_t, h_t, \eta_{t,i}$ can be obtained by, for $t \leq T$

$$C_t = -\frac{\alpha_{t,2} (E_{t+\Delta t} - A_{t+\Delta t}^2) r_{t,1}}{(E_{t+\Delta t} - A_{t+\Delta t}^2) \alpha_{k,2}^2 + E_{t+\Delta t} \sigma_{t,2}^2 \Delta t} \quad (3.40)$$

$$\eta_{t,i} = \frac{\alpha_{t,2} [(E_{t+\Delta t} - A_{t+\Delta t}^2) \alpha_{t,L_i} + A_{t+\Delta t} B_{t+\Delta t} r_{t,L_i}] + E_{t+\Delta t} D_{t,2L_i}}{(E_{t+\Delta t} - A_{t+\Delta t}^2) \alpha_{k,2}^2 + E_{t+\Delta t} \sigma_{t,2}^2 \Delta t} \quad (3.41)$$

$$h_t = \frac{\alpha_{t,2} \left[\left(\frac{1}{\gamma} A_{t+\Delta t} - \frac{1}{2} F_{t+\Delta t} \right) + M_{t+\Delta t} A_{t+\Delta t} \right]}{(E_{t+\Delta t} - A_{t+\Delta t}^2) \alpha_{k,2}^2 + E_{t+\Delta t} \sigma_{t,2}^2 \Delta t}. \quad (3.42)$$

Also, we have

Theorem 3.2.5.2. *For any $t \leq T$, function $g(t, X_t, \psi_t)$ and $f(t, X_t, \psi_t)$ can be obtained by*

$$g(t, X_t, \psi_t) = A_t X_t + \mathbf{B}_t^T \psi_{t,L} + M_t \quad (3.43)$$

$$f(t, X_t, \psi_t) = E_t X_t^2 + F_t X_t + \psi_t^T \mathbf{G}_t \psi_t + \mathbf{H}_t^T \psi_t + \mathbf{I}_t^T \psi_t X_t + N_t \quad (3.44)$$

and

$$E_t - A_t^2 \geq 0 \quad (3.45)$$

$$E_t > 0 \quad (3.46)$$

where coefficients can be iterated given by

$$A_t = A_{t+\Delta t}(r_{t,1} + \alpha_{t,2}\Delta t C_t) \quad (3.47)$$

$$A_T = 1 \quad (3.48)$$

$$B_{t,i} = A_{t+\Delta t}Q_{t,i} + B_{t+\Delta t,i}r_{t,Li} \quad (3.49)$$

$$B_{T,i} = 0 \quad (3.50)$$

$$M_t = M_{t+\Delta t} + A_{t+\Delta t}\alpha_{t,2}h_t \quad (3.51)$$

$$M_T = 0 \quad (3.52)$$

$$E_t = E_{t+\Delta t} [\sigma_{t,2}^2 \Delta t C_t^2 + (r_{t,1} + \alpha_{t,2}C_t)^2] \quad (3.53)$$

$$E_T = 1 \quad (3.54)$$

$$F_t = E_{t+\Delta t} [2\sigma_{t,2}^2 \Delta t h_t C_t + 2(r_{t,1} + \alpha_{t,2}C_t)\alpha_{t,2}h_t] + F_{t+\Delta t}(r_{t,1} + \alpha_{t,2}C_t) \quad (3.55)$$

$$F_T = 0 \quad (3.56)$$

$$\begin{aligned} G_{t,ij} = & E_{t+\Delta t}(\sigma_{t,2}^2 \Delta t \eta_{t,i} \eta_{t,j} - 2D_{t,2L_i} \eta_{t,j}) + E_{t+\Delta t} Q_{t,i} Q_{t,j} + G_{t+\Delta t} r_{t,L_i} r_{t,L_j} \\ & + (E_{t+\Delta t} + G_{t+\Delta t}) D_{t,L_i L_j} + I_{t+\Delta t,i} \bar{c}_{t,ij} \end{aligned} \quad (3.57)$$

$$G_T = 0 \quad (3.58)$$

$$\begin{aligned} H_{t,i} = & E_{t+\Delta t} [2\sigma_{t,2}^2 \Delta t \eta_{t,i} h_t - 2\sigma_{t,2}^2 \Delta t \theta_{t,i} h_t + 2\alpha_{t,2} Q_{t,i} h_t] + F_{t+\Delta t} Q_{t,i} + H_{t+\Delta t,i} r_{t,L_i} \\ & + I_{t+\Delta t,i} \bar{b}_{t,i} \end{aligned} \quad (3.59)$$

$$H_T = 0 \quad (3.60)$$

$$I_{t,i} = E_{t+\Delta t} [2\sigma_{t,2}^2 \Delta t \eta_{t,i} C_t - 2\sigma_{t,2}^2 \Delta t \theta_{t,i} C_t + 2(r_{t,1} + \alpha_{t,2}C_t)Q_{t,i}] + I_{t+\Delta t} \bar{a}_{t,i} \quad (3.61)$$

$$I_T = 0 \quad (3.62)$$

$$N_t = E_{t+\Delta t} [\sigma_{t,2}^2 \Delta t + \alpha_{t,2}^2] h_t^2 + F_{t+\Delta t} \alpha_{t,2} h_t + N_{t+\Delta t} \quad (3.63)$$

$$N_T = 0 \quad (3.64)$$

where $Q_{t,i} = (\alpha_{t,2}\eta_{t,i} - \alpha_{t,L_i})$, $\bar{a}_t, \bar{b}_t, \bar{c}_t$ can be obtained from equation (3.77), (3.78) and (3.79).

3.2.6 Proof of Theorem 3.2.5.1

Our proof relies on the backward induction. It is trivial that equation (3.43)-(3.46) hold for $t = T$ as in chapter 3.1.4. Assuming equation (3.39) and equations (3.43)-(3.46) hold for $t \geq k + \Delta t$, we now examine the case for $t = k$. Let $u = (u_k, u_{k+\Delta t}^*, \dots, u_T^*)$, from definition (3.37), (3.38) and the Tower Property, the utility function can be written as

$$\begin{aligned}
& J(k, X_k, \psi_k; u) \\
&= \mathbb{E}_{k, X_k, \psi_k} [X_T^u] - \frac{\gamma}{2} \text{Var}_{k, X_k, \psi_k} [X_T^u] \\
&= \mathbb{E}_{k, X_k, \psi_k} \left[\mathbb{E}_{k+\Delta t, X_{k+\Delta t}^{u_k}} [X_T^{u^*}] \right] \\
&\quad - \frac{\gamma}{2} \left(\mathbb{E}_{k, X_k, \psi_k} \left[\mathbb{E}_{k+\Delta t, X_{k+\Delta t}^{u_k}} \left[(X_T^{u^*})^2 \right] \right] - \left(\mathbb{E}_{k, X_k, \psi_k} \left[\mathbb{E}_{k+\Delta t, X_{k+\Delta t}^{u_k}} [X_T^{u^*}] \right] \right)^2 \right) \\
&= \mathbb{E}_{k, X_k, \psi_k} [g(k + \Delta t, X_{k+\Delta t}^{u_k})] \\
&\quad - \frac{\gamma}{2} \left(\mathbb{E}_{k, X_k, \psi_k} [f(k + \Delta t, X_{k+\Delta t}^{u_k})] - \left(\mathbb{E}_{k, X_k, \psi_k} [g(k + \Delta t, X_{k+\Delta t}^{u_k})] \right)^2 \right).
\end{aligned}$$

Using equation (3.43) and equation (3.44), we have

$$\begin{aligned}
& J(k, X_k, \psi_k; u) \\
&= A_{k+\Delta t} \mathbb{E}_{k, X_k, \psi_k} [X_{k+\Delta t}^{u_k}] + \sum_{i=1}^{n_L} B_{k+\Delta t, i} \mathbb{E}_{k, X_k, \psi_k} [\psi_{k+\Delta t, L_i}] + \mathbb{E}_{k, X_k, \psi_k} [M_{k+\Delta t}] \\
&\quad - \frac{\gamma}{2} \left\{ E_{k+\Delta t} \mathbb{E}_{k, X_k, \psi_k} \left[(X_{k+\Delta t}^{u_k})^2 \right] + F_{k+\Delta t} \mathbb{E}_{k, X_k, \psi_k} [X_{k+\Delta t}^{u_k}] \right. \\
&\quad + \sum_{i,j}^{n_L} G_{k+\Delta t, ij} \mathbb{E}_{k, X_k, \psi_k} [\psi_{k+\Delta t, L_i} \psi_{k+\Delta t, L_j}] \\
&\quad + \sum_{i=1}^{n_L} H_{k+\Delta t, i} \mathbb{E}_{k, X_k, \psi_k} [\psi_{k+\Delta t, L_i}] + \sum_{i=1}^{n_L} I_{k+\Delta t, i} \mathbb{E}_{k, X_k, \psi_k} [X_{k+\Delta t}^{u_k} \psi_{k+\Delta t, L_i}] + N_{k+\Delta t} \\
&\quad \left. - \left(A_{k+\Delta t} \mathbb{E}_{k, X_k, \psi_k} [X_{k+\Delta t}^{u_k}] + \sum_{i=1}^{n_L} B_{k+\Delta t, i} \mathbb{E}_{k, X_k, \psi_k} [\psi_{k+\Delta t, L_i}] \right. \right. \\
&\quad \left. \left. + \mathbb{E}_{k, X_k, \psi_k} [M_{k+\Delta t}] \right)^2 \right\} \\
&= -\frac{\gamma}{2} q_{k,1} (\psi_{k,2} u_{k,2})^2 + q_{k,2} (\psi_{k,2} u_{k,2}) + q_{k,3}
\end{aligned}$$

where $q_{k,1}, q_{k,2}, q_{k,3}$ are coefficients independent of $\psi_k u_k$ and are given as follows

$$q_{k,1} = (E_{k+\Delta t} - A_{k+\Delta t}^2) \alpha_{k,2}^2 + E_{k+\Delta t} \sigma_{k,2}^2 \Delta t \quad (3.65)$$

$$\begin{aligned}
q_{k,2} = & \alpha_{k,2} \left[\left(A_{k+\Delta t} - \frac{\gamma}{2} F_{k+\Delta t} \right) - \gamma (E_{k+\Delta t} - A_{k+\Delta t}^2) \left(r_{k,1} X_k - \sum_{i=1}^{n_L} \alpha_{k,L_i} \psi_{k,L_i} \right) \right. \\
& \left. + A_{k+\Delta t} \sum_{i=1}^{n_L} \gamma B_{k+\Delta t, i} r_{k,L_i} \psi_{k,L_i} + \gamma M_{k+\Delta t} A_{k+\Delta t} \right] + \gamma E_{k+\Delta t} \sum_{i=1}^{n_L} D_{k,2L_i} \psi_{k,L_i} \\
& \quad (3.66)
\end{aligned}$$

$$\begin{aligned}
q_{k,3} = & -\frac{\gamma}{2} \sum_{i,j=1}^{n_L} \left[G_{k+\Delta t,ij} r_{k,L_i} r_{k,L_j} + (G_{k+\Delta t,ij} + E_{k+\Delta t}) D_{k,L_i L_j} + I_{k+\Delta t,i} \bar{c}_{k,ij} \right] \psi_{k,L_i} \psi_{k,L_j} \\
& + \sum_{i=1}^{n_L} \left(B_{k+\Delta t,i} - \frac{\gamma}{2} H_{k+\Delta t,i} \right) r_{k,L_i} \psi_{k,L_i} - \frac{\gamma}{2} \sum_{i=1}^{n_L} I_{k+\Delta t,i} [\bar{a}_{k,i} X_k \psi_{k,L_i} + \bar{b}_{k,i} \psi_{k,L_i}] \\
& + \left(A_{k+\Delta t} - \frac{\gamma}{2} F_{k+\Delta t} \right) \left(r_{k,1} X_k - \sum_{i=1}^{n_L} \alpha_{k,L_i} \psi_{k,L_i} \right) \\
& - \frac{\gamma}{2} E_{k+\Delta t} \left(r_{k,1} X_k - \sum_{i=1}^{n_L} \alpha_{k,L_i} \psi_{k,L_i} \right)^2 + M_{k+\Delta t} - \frac{\gamma}{2} N_{k+\Delta t} \\
& + \frac{\gamma}{2} \left[A_{k+\Delta t} \left(r_{k,1} X_k - \sum_{i=1}^{n_L} \alpha_{k,L_i} \psi_{k,L_i} \right) + \sum_{i=1}^{n_L} B_{k+\Delta t,i} r_{k,L_i} \psi_{k,L_i} + M_{k+\Delta t} \right]^2
\end{aligned} \tag{3.67}$$

Thus the utility function becomes a function quadratic in $\psi_{k,2} u_{k,2}$. From the assumption in equation (3.45), (3.46), the quadratic function is strictly concave in $\psi_{k,2} u_{k,2}$. Hence, we can obtain the control $u_{k,2}^*$ by maximising the quadratic function. By the first condition, the quadratic function is maximised by adopting

$$u_{k,2}^* = \frac{1}{\psi_{k,2}} \frac{q_{k,2}}{\gamma q_{k,1}}. \tag{3.68}$$

The control u^* can also be written into

$$u_{k,2}^* = \frac{1}{\psi_{k,2}} (C_k X_k + h_k + \boldsymbol{\eta}_k^T \boldsymbol{\psi}_{k,L})$$

as in equation (3.39). Hence equation (3.39) holds for $t = k$ and we complete the proof of Theorem 3.2.5.1.

3.2.7 Proof of Theorem 3.2.5.2

From equation (3.2), we can determine the expected wealth and the second moment at time $t + \Delta t$ based on time t as

$$\mathbb{E}_{t, X_t, \psi_t}[X_{t+\Delta t}^u] = \sum_i [(1 + \mu_{t,i}\Delta t)\psi_{t,i}]u_{t,i} \quad (3.69)$$

$$= \alpha_{t,2}\psi_{t,2}u_{t,2} + r_{t,1}X_t - \sum_{L_i} \alpha_{t,L_i}\psi_{t,L_i} \quad (3.70)$$

The variance of wealth at time $t + \Delta t$ in terms of those on time t is

$$Var_{t, X_t, \psi_t}[X_{t+\Delta t}^u] = \sum_{i,j} \sigma_{t,i}\sigma_{t,j}\rho_{ij}\Delta t\psi_{t,i}u_{t,i}\psi_{t,j}u_{t,j} \quad (3.71)$$

$$\begin{aligned} &= \sigma_{t,2}^2\Delta t(\psi_{t,2}u_{t,2})^2 - 2\sum_{L_i} D_{t,2L_i}\psi_{t,L_i}\psi_{t,2}u_{t,2} \\ &\quad + \sum_{L_i, L_j} D_{t,L_i L_j}\psi_{t,L_i}\psi_{t,L_j} \end{aligned} \quad (3.72)$$

Thus we have

$$\begin{aligned} \mathbb{E}_{t, X_t, \psi_t} \left[(X_{t+\Delta t}^u)^2 \right] &= (\sigma_{t,2}^2\Delta t + \alpha_{t,2}^2)(\psi_{t,2}u_{t,2})^2 \\ &\quad + \left[2\alpha_{t,2} \left(r_{t,1}X_t - \sum_{i=1}^{n_L} \alpha_{t,L_i}\psi_{t,L_i} \right) - 2\sum_{i=1}^{n_L} D_{t,2L_i}\psi_{t,L_i} \right] (\psi_{t,2}u_{t,2}) \\ &\quad + \left(r_{t,1}X_t - \sum_{i=1}^{n_L} \alpha_{t,L_i}\psi_{t,L_i} \right)^2 + \sum_{i,j=1}^{n_L} D_{t,L_i L_j}\psi_{t,L_i}\psi_{t,L_j} \end{aligned} \quad (3.73)$$

We can also determine the expected price of asset i at time $t + \Delta t$ based on those at time t as

$$\mathbb{E}_{t, X_t, \psi_t} [\psi_{t+\Delta t, i}^u] = r_{t, i} \psi_{t, i}, \quad (3.74)$$

$$\mathbb{E}_{t, X_t, \psi_t} [\psi_{t+\Delta t, i} \psi_{t+\Delta t, j}] = (r_{t, i} r_{t, j} + D_{t, ij}) \psi_{t, i} \psi_{t, j}, \quad (3.75)$$

$$\mathbb{E}_{t, X_t, \psi_t} [X_{t+\Delta t} \psi_{t+\Delta t, i}] = \bar{a}_{t, i} X_t \psi_{t, L_i} + \bar{b}_{t, i} \psi_{t, L_i} + \sum_{j=1}^{n_L} \bar{c}_{t, ij} \psi_{t, L_i} \psi_{t, L_j}, \quad (3.76)$$

where

$$\bar{a}_{t, i} = r_{t, 1} r_{t, L_i} + (\alpha_{t, 2} r_{t, L_i} + D_{t, 2L_i}) C_t, \quad (3.77)$$

$$\bar{b}_{t, i} = (\alpha_{t, 2} r_{t, L_i} + D_{t, 2L_i}) h_t, \quad (3.78)$$

$$\bar{c}_{t, ij} = (\alpha_{t, 2} \eta_{t, j} - \alpha_{t, L_j}) r_{t, L_i} + (D_{t, 2L_i} \eta_{t, L_j} - D_{t, L_i L_j}). \quad (3.79)$$

We now prove Theorem 3.2.5.2 using the backward induction. It is trivial that equation (3.43)-(3.46) hold for $t = T$ as in chapter 3.1.4. Assuming equation (3.39) and equations (3.43)-(3.46) hold for $t \geq k + \Delta t$, we seek to check the case for $t = k$. Let $u = (u_k, u_{k+\Delta t}^*, \dots, u_T^*)$, from definition (3.37), (3.38) and the Tower Property, we

have

$$\begin{aligned}
f(k, X_k, \psi_k) &= \mathbb{E}_{k, X_k, \psi_k} [X_T^u] \\
&= \mathbb{E}_{k, X_k, \psi_k} \left[\mathbb{E}_{k+\Delta t, X_{k+\Delta t}^{u_k}, \psi_{k+\Delta t}^{u_k}} [X_T^{u*}] \right] \\
&= \mathbb{E}_{k, X_k, \psi_k} [g(k + \Delta t, X_{k+\Delta t}^{u_k}, \psi_{k+\Delta t}^{u_k})] \\
&= A_{k+\Delta t} \mathbb{E}_{k, X_k, \psi_k} [X_{k+\Delta t}^{u_k}] + \sum_{i=1}^{n_L} B_{k+\Delta t, i} \mathbb{E}_{k, X_k, \psi_k} [\psi_{k+\Delta t, i}] + M_{k+\Delta t} \\
&= A_{k+\Delta t} \left[(r_{k,1} + \alpha_{k,2} C_k) X_k + \sum_{i=1}^{n_L} Q_{k,i} \psi_{k, L_i} + \alpha_{k,2} h_k \right] \\
&\quad + \sum_{i=1}^{n_L} B_{k+\Delta t, i} r_{t, L_i} \psi_{k, L_i} + M_{k+\Delta t} \\
&= A_k X_k + \mathbf{B}_k^T \boldsymbol{\psi}_{k, L} + M_k
\end{aligned}$$

$$\begin{aligned}
g(k, X_k, \psi_k) &= \mathbb{E}_{k, X_k, \psi_k} [(X_T^u)^2] \\
&= \mathbb{E}_{k, X_k, \psi_k} \left[\mathbb{E}_{k+\Delta t, X_{k+\Delta t}^{u_k}, \psi_{k+\Delta t}^{u_k}} [(X_T^{u*})^2] \right] \\
&= \mathbb{E}_{k, X_k, \psi_k} [f(k + \Delta t, X_{k+\Delta t}^{u_k}, \psi_{k+\Delta t}^{u_k})] \\
&= E_{k+\Delta t} \mathbb{E}_{k, X_k, \psi_k} [(X_{k+\Delta t}^{u_k})^2] + F_{k+\Delta t} \mathbb{E}_{k, X_k, \psi_k} [X_{k+\Delta t}^{u_k}] \\
&\quad + \sum_{i,j}^{n_L} G_{k+\Delta t, ij} \mathbb{E}_{k, X_k, \psi_k} [\psi_{k+\Delta t, L_i} \psi_{k+\Delta t, L_j}] \\
&\quad + \sum_{i=1}^{n_L} H_{k+\Delta t, i} \mathbb{E}_{k, X_k, \psi_k} [\psi_{k+\Delta t, L_i}] + \sum_{i=1}^{n_L} I_{k+\Delta t, i} \mathbb{E}_{k, X_k, \psi_k} [X_{k+\Delta t}^{u_k} \psi_{k+\Delta t, L_i}] \\
&\quad + N_{k+\Delta t} \\
&= E_k X_k^2 + F_k X_k + \boldsymbol{\psi}_k^T \mathbf{G}_k \boldsymbol{\psi}_k + \mathbf{H}_k^T \boldsymbol{\psi}_k + \mathbf{I}_k^T \boldsymbol{\psi}_k X_k + N_k
\end{aligned}$$

Hence, equation (3.43) (3.44) holds for $t = k$.

Also, $E_{k+\Delta t} - A_{k+\Delta t}^2 \geq 0$ and $E_{k+\Delta t} > 0$ implies

$$E_k - A_k^2 = (E_{k+\Delta t} - A_{k+\Delta t}^2) (r_{k,1} + \alpha_{k,2}C_k)^2 + E_{k+\Delta t}\sigma_{k,2}^2\Delta t C_k^2 \geq 0 \quad (3.80)$$

$$E_k = E_{k+\Delta t} [\sigma_{k,2}^2\Delta t C_k^2 + (r_{k,1} + \alpha_{k,2}C_k)^2] > 0 \quad (3.81)$$

Therefore, equation (3.45) and equation (3.46) hold for $t = k$. This complete the prove for Theorem 3.2.5.2.

4 Mean-Variance Asset-Liability Portfolio Selection with Jumps

In chapter 3, we studied the Mean-Variance Asset-Liability portfolio selection problem assuming the assets and liabilities followed a general linear SDE. In this chapter, we shall further extend the results to models with jumps. Similar to Corollary 3.1.2.1, we shall investigate the recursive relation assuming that the linearity of prices in equation (3.5) and equation (3.6) holds. In section 4.1, we extend the previous setting in (3.29) to models by adding a jump component, which is formulated as a compound Poisson process. In section 4.4, we find the optimal solution for the jump model under mean-variance approach.

4.1 Problem Setting

Given the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Denote \mathbb{E} as the expectation with respect to probability measure \mathbb{P} . All the financial assets and liabilities in the market are given

by, at any time $t \in \tau := \{0, \Delta t, \dots, T\}$,

$$\begin{aligned}
\psi_{t+\Delta t} = & \begin{pmatrix} 1 + \mu_{t,1}\Delta t & 0 & \cdots & 0 \\ 0 & 1 + \mu_{t,2}\Delta t & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 + \mu_{t,L_{n_L}}\Delta t \end{pmatrix} \begin{pmatrix} \psi_{t,1} \\ \psi_{t,2} \\ \vdots \\ \psi_{t,L_{n_L}} \end{pmatrix} \\
& + \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & \sigma_{t,2}\Delta W_t & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_{t,L_{n_L}}\Delta W_t \end{pmatrix} \begin{pmatrix} \psi_{t,1} \\ \psi_{t,2} \\ \vdots \\ \psi_{t,L_{n_L}} \end{pmatrix} \\
& + \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & Y_{t,2}\Delta N_t & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & Y_{t,L_{n_L}}\Delta N_t \end{pmatrix} \begin{pmatrix} \psi_{t,1} \\ \psi_{t,2} \\ \vdots \\ \psi_{t,L_{n_L}} \end{pmatrix}, \quad (4.1)
\end{aligned}$$

where W_t is a vector of standard Brownian motion with drift μ_t and volatility σ_t and ρ_{ij} denotes the correlation between two Brownian motions W_i, W_j ; Y_t is a sequence of i.i.d. random variables with mean a and variance b^2 , denoting the size of jumps; N_t is a Poisson process with intensity λ and N_t is independent with any other Poisson process. We always assume that Y_t, W_t and N_t are independent. For small interval Δt , ΔN_t denotes the number of jumps occurring in time interval $(t, t + \Delta t]$, with probability

$$\mathbb{P}[\Delta N_t = 0] = e^{-\lambda\Delta t} \approx 1, \quad (4.2)$$

$$\mathbb{P}[\Delta N_t = 1] = \lambda\Delta t e^{-\lambda\Delta t} \approx 0, \quad (4.3)$$

$$\mathbb{P}[\Delta N_t = k] = \frac{e^{-\lambda\Delta t} (\lambda\Delta t)^k}{k!}, \quad (4.4)$$

mean and variance

$$\mathbb{E}[\Delta N_t] = \lambda \Delta t,$$

$$Var[\Delta N_t] = \lambda \Delta t.$$

4.2 Wealth Process

Assuming that in the market, we have one riskless asset, one risky asset and n_L liabilities. We denote $\psi_{t,1}$ as the riskless asset with interest rate $\mu_{t,1}$ and random terms $\sigma_{t,1} = Y_{t,1} = 0$; $\psi_{t,2}$ as the risky asset with appreciate rate $\mu_{t,2}$ and volatility $\sigma_{t,2}$; ψ_{t,L_i} as liability with appreciate rate $\mu_{t,2}$ and volatility $\sigma_{t,2}$. The jump process of risky asset ψ_{t,L_i} and liabilities ψ_{t,L_i} are governed by $Y_{t,2}\Delta N_t$ and $Y_{t,L_i}\Delta N_t$ respectively. Denote $\mathbf{u}_t = (u_{t,1}, u_{t,2}, -1, \dots, -1)^T$ as the shares of holding at time t . We further define wealth process X_t as the total assets minus total liabilities. In particular, for liabilities L_i , we always assume $u_{t,L_i} = -1$. At any time $t \in \{0, \Delta t, \dots, T\}$, we can obtain the total wealth via

$$X_t = \psi_t^T \mathbf{u}_t. \quad (4.5)$$

4.3 Mean Variance Optimisation Problem

The purpose of an investor is to find the optimal investment strategy by maximising the following utility function:

$$J(t, X_t, \psi_t; u) := \mathbb{E}_{t, X_t, \psi_t}[X_T^u] - \frac{\gamma}{2} Var_{t, X_t, \psi_t}[X_T^u], \quad (4.6)$$

where γ is a risk averse parameter; $\mathbb{E}_{t, X_t, \psi_t}[\cdot]$ and $Var_{t, X_t, \psi_t}[\cdot]$ denote the mean and variance of the terminal wealth X_T conditioned on the current wealth X_t and current liabilities ψ_t , i.e. $\mathbb{E}_{t, X_t, \psi_t}[X_T^u] = \mathbb{E}[X_N^u | X_t, \psi_t]$, $Var_{t, X_t, \psi_t}[X_T^u] = Var[X_N^u | X_t, \psi_t]$.

Then the target of an investor is to find the optimal strategy u_t in order to maximise the objective function (4.6), subjected to the current condition on current total wealth in equation (4.5).

4.4 Optimal Solution

Define

$$g(t, X_t, \psi_t) := \mathbb{E}_{t, X_t, \psi_t} [X_T^{u*}], \quad (4.7)$$

$$f(t, X_t, \psi_t) := \mathbb{E}_{t, X_t, \psi_t} [(X_T^{u*})^2]. \quad (4.8)$$

Theorem 4.4.1. *The optimal control, u_t^* , for the objective function (4.6) subjected to the wealth process (4.5), can be determined as:*

$$u_{t,2}^* = \frac{1}{\psi_{t,2}} (C_t X_t + \boldsymbol{\eta}_t^T \boldsymbol{\psi}_{t,L} + h_t) \quad (4.9)$$

where coefficients $C_t, h_t, \eta_{t,i}$ can be obtained by, for $t \leq T$,

$$C_t = - \frac{(\alpha_{k,2} + \beta_2) (E_{t+\Delta t} - A_{t+\Delta t}^2) r_{t,1}}{(E_{t+\Delta t} - A_{t+\Delta t}^2) (\alpha_{k,2} + \beta_2)^2 + E_{t+\Delta t} [\sigma_{t,2}^2 \Delta t + (a_2^2 + b_2^2) \lambda_2 \Delta t]}, \quad (4.10)$$

$$\eta_{t,i} = \frac{(\alpha_{k,2} + \beta_2) [(E_{t+\Delta t} - A_{t+\Delta t}^2) (\alpha_{t,L_i} + \beta_{L_i}) + A_{t+\Delta t} B_{t+\Delta t} (r_{t,L_i} + \beta_{L_i})] + E_{t+\Delta t} D_{t,2L_i}}{(E_{t+\Delta t} - A_{t+\Delta t}^2) (\alpha_{k,2} + \beta_2)^2 + E_{t+\Delta t} [\sigma_{t,2}^2 \Delta t + (a_2^2 + b_2^2) \lambda_2 \Delta t]}, \quad (4.11)$$

$$h_t = \frac{(\alpha_{k,2} + \beta_2) \left[\left(\frac{1}{\gamma} A_{t+\Delta t} - \frac{1}{2} F_{t+\Delta t} \right) + M_{t+\Delta t} A_{t+\Delta t} \right]}{(E_{t+\Delta t} - A_{t+\Delta t}^2) (\alpha_{k,2} + \beta_2)^2 + E_{t+\Delta t} [\sigma_{t,2}^2 \Delta t + (a_2^2 + b_2^2) \lambda_2 \Delta t]}. \quad (4.12)$$

Also, we have

Theorem 4.4.2. *For any $t \leq T$, function $g(t, X_t, \psi_t)$ and $f(t, X_t, \psi_t)$ can be obtained by*

$$g(t, X_t, \psi_t) = A_t X_t + \mathbf{B}_t^T \psi_{t,L} + M_t, \quad (4.13)$$

$$f(t, X_t, \psi_t) = E_t X_t^2 + F_t X_t + \psi_t^T \mathbf{G}_t \psi_t + \mathbf{H}_t^T \psi_t + \mathbf{I}_t^T \psi_t X_t + N_t, \quad (4.14)$$

and

$$E_t - A_t^2 \geq 0, \quad (4.15)$$

$$E_t > 0. \quad (4.16)$$

where coefficients can be iterated obtained by

$$A_t = A_{t+\Delta t} [r_{t,1} + (\alpha_{t,2} + \beta_2)C_t] \quad (4.17)$$

$$A_T = 1 \quad (4.18)$$

$$B_{t,i} = A_{t+\Delta t} [(\alpha_{t,2} + \beta_2)\eta_{t,i} - (\alpha_{L_i} + \beta_{L_i})] + B_{t+\Delta t,i}(r_{t,L_i} + \beta_{L_i}) \quad (4.19)$$

$$B_{T,i} = 0 \quad (4.20)$$

$$M_t = A_{t+\Delta t}(\alpha_{t,2} + \beta_2)h_t + M_{t+\Delta t} \quad (4.21)$$

$$M_T = 0 \quad (4.22)$$

$$E_t = E_{t+\Delta t} \left[\left(\sigma_{t,2}^2 \Delta t + D'_{22} + \alpha_{t,2}^2 + 2\alpha_{t,2}\beta_2 \right) C_t^2 + 2(\alpha_{t,2} + \beta_2)r_{t,1}C_t + r_{t,1}^2 \right] \quad (4.23)$$

$$E_T = 1 \quad (4.24)$$

$$F_t = E_{t+\Delta t} \cdot 2h_t \left[\left(\sigma_{t,2}^2 \Delta t + D'_{22} + \alpha_{t,2}^2 + 2\alpha_{t,2}\beta_2 \right) C_t + (\alpha_{t,2} + \beta_2)r_{t,1} \right] \\ + F_{t+\Delta t} [r_{t,1} + (\alpha_{t,2} + \beta_2)C_t] \quad (4.25)$$

$$F_T = 0 \quad (4.26)$$

$$G_{t,ij} = E_{t+\Delta t} \left[\left(\sigma_{t,2}^2 \Delta t + D'_{22} + \alpha_{t,2}^2 + 2\alpha_{t,2}\beta_2 \right) \eta_{t,i}\eta_{t,j} - 2(\alpha_{t,2} + \beta_2)(\alpha_{t,L_i} + \beta_{L_i})\eta_{t,j} \right. \\ \left. - 2D_{t,2L_i}\eta_{t,j} + (\alpha_{t,L_i} + \beta_{L_i})(\alpha_{t,L_j} + \beta_{L_j}) + (D_{t,L_iL_j} + D'_{L_iL_j} - \beta_{L_i}\beta_{L_j}) \right] \\ + G_{t+\Delta t} \left(r_{t,L_i}r_{t,L_j} + \beta_{L_i}r_{t,L_j} + \beta_{L_j}r_{t,L_i} + D_{t,L_iL_j} + D'_{L_iL_j} \right) + I_{t+\Delta t,i}\bar{c}_{t,ij} \quad (4.27)$$

$$G_T = 0 \quad (4.28)$$

$$\begin{aligned}
H_{t,i} = & E_{t+\Delta t} \cdot 2h_t \left[\left(\sigma_{t,2}^2 \Delta t + D'_{22} + \alpha_{t,2}^2 + 2\alpha_{t,2}\beta_2 \right) - (\alpha_{t,2} + \beta_2) (\alpha_{t,L_i} + \beta_{L_i}) - D_{t,2L_i} \right] \\
& + F_{t+\Delta t} [(\alpha_{t,2} + \beta_2) \eta_{t,i} - (\alpha_{t,L_i} + \beta_{L_i})] + H_{t+\Delta t,i} (r_{t,L_i} + \beta_{L_i}) + I_{t+\Delta t,i} \bar{b}_{t,i}
\end{aligned} \tag{4.29}$$

$$H_T = 0 \tag{4.30}$$

$$\begin{aligned}
I_{t,i} = & E_{t+\Delta t} \left[2 \left(\sigma_{t,2}^2 \Delta t + D'_{22} + \alpha_{t,2}^2 + 2\alpha_{t,2}\beta_2 \right) C_t \eta_{t,i} - 2(\alpha_{t,2} + \beta_2) [(\alpha_{t,L_i} + \beta_{L_i}) C_t \right. \\
& \left. - r_{t,1} \eta_{t,i}] - 2C_t D_{t,2L_i} - 2r_{t,1} (\alpha_{t,L_i} + \beta_{L_i}) \right] + I_{t+\Delta t,i} \bar{a}_{t,i}
\end{aligned} \tag{4.31}$$

$$I_T = 0 \tag{4.32}$$

$$\begin{aligned}
N_t = & E_{t+\Delta t} \left(\sigma_{t,2}^2 \Delta t + D'_{22} + \alpha_{t,2}^2 + 2\alpha_{t,2}\beta_2 \right) h_t^2 + F_{t+\Delta t} (\alpha_{t,2} + \beta_2) h_t + N_{t+\Delta t}
\end{aligned} \tag{4.33}$$

$$N_T = 0 \tag{4.34}$$

where $r_{t,i} = (1 + \mu_{t,i} \Delta t)$ denotes the fixed growth on the i th asset; $\alpha_{t,i} = (\mu_{t,i} - \mu_{t,1}) \Delta t$ denotes the net interest between i th asset and riskless asset; $\beta_i = \mathbb{E}[Y_i \Delta N_t] = a_i \lambda_i \Delta t$ denotes the expected value of the jump size for asset i ; $D_{t,ij} = \sigma_{t,i} \sigma_{t,j} \rho_{ij} \Delta t$ denotes the correlation of i th and j th asset generated by the Brownian motion; D'_{ij} denotes the correlation of i th and j th asset generated by the Poisson jumps, which is given as $(a_i^2 + b_i^2) \lambda_i \Delta t + (a_i \lambda_i \Delta t)^2$ for $i = j$ and $a_i \lambda_i \Delta t a_j \lambda_j \Delta t$ for $i \neq j$; $\bar{a}_t, \bar{b}_t, \bar{c}_t$ can be known from equation (4.45)(4.46) and (4.47).

4.5 Proof of Optimal Solution

In this section, we shall prove Theorem 4.4.2 and Theorem 4.4.1. Our proof relies on the backward induction.

4.5.1 Proof of Theorem 4.4.1

It is trivial that equation (4.13)-(4.16) hold for $t = T$. Assuming equation (4.9) and equations (4.13)-(4.16) hold for $t \geq k + \Delta t$, we are now checking the case for $t = k$. Let $u = (u_k, u_{k+\Delta t}^*, \dots, u_T^*)$, from definition (4.7), (4.8) and the Tower Property, the utility function can be written as

$$\begin{aligned}
& J(k, X_k, \psi_k; u) \\
&= \mathbb{E}_{k, X_k, \psi_k} [X_T^u] - \frac{\gamma}{2} \text{Var}_{k, X_k, \psi_k} [X_T^u] \\
&= \mathbb{E}_{k, X_k, \psi_k} \left[\mathbb{E}_{k+\Delta t, X_{k+\Delta t}^{u_k}} [X_T^{u^*}] \right] \\
&\quad - \frac{\gamma}{2} \left(\mathbb{E}_{k, X_k, \psi_k} \left[\mathbb{E}_{k+\Delta t, X_{k+\Delta t}^{u_k}} \left[(X_T^{u^*})^2 \right] \right] - \left(\mathbb{E}_{k, X_k, \psi_k} \left[\mathbb{E}_{k+\Delta t, X_{k+\Delta t}^{u_k}} [X_T^{u^*}] \right] \right)^2 \right) \\
&= \mathbb{E}_{k, X_k, \psi_k} \left[g(k + \Delta t, X_{k+\Delta t}^{u_k}) \right] \\
&\quad - \frac{\gamma}{2} \left(\mathbb{E}_{k, X_k, \psi_k} \left[f(k + \Delta t, X_{k+\Delta t}^{u_k}) \right] - \left(\mathbb{E}_{k, X_k, \psi_k} \left[g(k + \Delta t, X_{k+\Delta t}^{u_k}) \right] \right)^2 \right).
\end{aligned}$$

Using equation (3.43) and equation (3.44), we have

$$\begin{aligned}
& J(k, X_k, \psi_k; u) \\
&= A_{k+\Delta t} \mathbb{E}_{k, X_k, \psi_k} [X_{k+\Delta t}^{u_k}] + \sum_{i=1}^{n_L} B_{k+\Delta t, i} \mathbb{E}_{k, X_k, \psi_k} [\psi_{k+\Delta t, L_i}] + \mathbb{E}_{k, X_k, \psi_k} [M_{k+\Delta t}] \\
&\quad - \frac{\gamma}{2} \left\{ E_{k+\Delta t} \mathbb{E}_{k, X_k, \psi_k} \left[(X_{k+\Delta t}^{u_k})^2 \right] + F_{k+\Delta t} \mathbb{E}_{k, X_k, \psi_k} [X_{k+\Delta t}^{u_k}] \right. \\
&\quad + \sum_{i,j}^{n_L} G_{k+\Delta t, ij} \mathbb{E}_{k, X_k, \psi_k} [\psi_{k+\Delta t, L_i} \psi_{k+\Delta t, L_j}] + \sum_{i=1}^{n_L} H_{k+\Delta t, i} \mathbb{E}_{k, X_k, \psi_k} [\psi_{k+\Delta t, L_i}] \\
&\quad + \sum_{i=1}^{n_L} I_{k+\Delta t, i} \mathbb{E}_{k, X_k, \psi_k} [X_{k+\Delta t}^{u_k} \psi_{k+\Delta t, L_i}] + N_{k+\Delta t} \\
&\quad \left. - \left(A_{k+\Delta t} \mathbb{E}_{k, X_k, \psi_k} [X_{k+\Delta t}^{u_k}] + \sum_{i=1}^{n_L} B_{k+\Delta t, i} \mathbb{E}_{k, X_k, \psi_k} [\psi_{k+\Delta t, L_i}] \right. \right. \\
&\quad \left. \left. + \mathbb{E}_{k, X_k, \psi_k} [M_{k+\Delta t}] \right)^2 \right\} \\
&= -\frac{\gamma}{2} q_{k,1} (\psi_{k,2} u_{k,2})^2 + q_{k,2} (\psi_{k,2} u_{k,2}) + q_{k,3},
\end{aligned}$$

where $q_{k,1}, q_{k,2}, q_{k,3}$ are coefficients independent of $\psi_k u_k$ and are given as follows

$$q_{k,1} = (E_{k+\Delta t} - A_{k+\Delta t}^2) (\alpha_{k,2} + \beta_2)^2 + E_{k+\Delta t} [\sigma_{k,2}^2 \Delta t + (a_2^2 + b_2^2) \lambda_2 \Delta t], \quad (4.35)$$

$$\begin{aligned} q_{k,2} = & (\alpha_{k,2} + \beta_2) \left[\left(A_{k+\Delta t} - \frac{\gamma}{2} F_{k+\Delta t} \right) \right. \\ & - \gamma (E_{k+\Delta t} - A_{k+\Delta t}^2) \left(r_{k,1} X_k - \sum_{i=1}^{n_L} (\alpha_{k,L_i} + \beta_{L_i}) \psi_{k,L_i} \right) \\ & + A_{k+\Delta t} \sum_{i=1}^{n_L} \gamma B_{k+\Delta t,i} (r_{k,L_i} + \beta_{L_i}) \psi_{k,L_i} + \gamma M_{k+\Delta t} A_{k+\Delta t} \left. \right] \\ & + \gamma E_{k+\Delta t} \sum_{i=1}^{n_L} D_{k,2L_i} \psi_{k,L_i}, \end{aligned} \quad (4.36)$$

$$\begin{aligned} q_{k,3} = & -\frac{\gamma}{2} \sum_{i,j=1}^{n_L} \left[G_{k+\Delta t,ij} (r_{k,L_i} r_{k,L_j} + a_{k,L_i} r_{k,L_i} + a_{k,L_j} r_{k,L_j}) - E_{k+\Delta t} \beta_{L_i} \beta_{L_j} \right. \\ & + (G_{k+\Delta t,ij} + E_{k+\Delta t}) (D_{k,L_i L_j} + D'_{L_i L_j}) + I_{k+\Delta t,i} \bar{c}_{k,ij} \left. \right] \psi_{k,L_i} \psi_{k,L_j} \\ & + \sum_{i=1}^{n_L} \left(B_{k+\Delta t,i} - \frac{\gamma}{2} H_{k+\Delta t,i} \right) (r_{k,L_i} + \beta_{L_i}) \psi_{k,L_i} \\ & - \frac{\gamma}{2} \sum_{i=1}^{n_L} I_{k+\Delta t,i} [\bar{a}_{k,i} X_k \psi_{k,L_i} + \bar{b}_{k,i} \psi_{k,L_i}] \\ & + \left(A_{k+\Delta t} - \frac{\gamma}{2} F_{k+\Delta t} \right) \left[r_{k,1} X_k - \sum_{i=1}^{n_L} (\alpha_{k,L_i} + \beta_{L_i}) \psi_{k,L_i} \right] \\ & - \frac{\gamma}{2} E_{k+\Delta t} \left[r_{k,1} X_k - \sum_{i=1}^{n_L} (\alpha_{k,L_i} + \beta_{L_i}) \psi_{k,L_i} \right]^2 + M_{k+\Delta t} - \frac{\gamma}{2} N_{k+\Delta t} \\ & + \frac{\gamma}{2} \left\{ A_{k+\Delta t} \left[r_{k,1} X_k - \sum_{i=1}^{n_L} (\alpha_{k,L_i} + \beta_{L_i}) \psi_{k,L_i} \right] + \sum_{i=1}^{n_L} B_{k+\Delta t,i} (r_{k,L_i} + \beta_{L_i}) \psi_{k,L_i} \right. \\ & \left. + M_{k+\Delta t} \right\}^2. \end{aligned} \quad (4.37)$$

Thus the utility function becomes a function quadratic in $\psi_{k,2} u_{k,2}$. From the assumption in equation (4.15), (4.16), the quadratic function is strictly concave in $\psi_{k,2} u_{k,2}$. Hence, we can obtain the control $u_{k,2}^*$ by maximising the quadratic function. By the

first derivative condition, the quadratic function is maximised by adopting

$$u_{k,2}^* = \frac{1}{\psi_{k,2}} \frac{q_{k,2}}{\gamma q_{k,1}}. \quad (4.38)$$

The control u^* can also be written into

$$u_{k,2}^* = \frac{1}{\psi_{k,2}} (C_k X_k + h_k + \boldsymbol{\eta}_k^T \boldsymbol{\psi}_{k,L})$$

as in equation (4.9). Hence equation (4.9) holds for $t = k$. So the proof of Theorem 4.4.1 is completed.

4.5.2 Proof of Theorem 4.4.2

We can determine expected wealth at time $t + \Delta t$ based on time those at t as

$$\mathbb{E}_{t, X_t, \psi_t} [X_{t+\Delta t}^u] = \sum_i [(1 + \mu_{t,i} \Delta t + \beta_i) \psi_{t,i}] u_{t,i} \quad (4.39)$$

$$= (\alpha_{t,2} + \beta_2) \psi_{t,2} u_{t,2} + r_{t,1} X_t - \sum_{L_i} (\alpha_{t,L_i} + \beta_{L_i}) \psi_{t,L_i}, \quad (4.40)$$

$$\begin{aligned} \mathbb{E}_{t, X_t, \psi_t} \left[(X_{t+\Delta t}^u)^2 \right] &= (\sigma_{t,2}^2 \Delta t + D'_{22} + \alpha_{t,2}^2 + 2\alpha_{t,2}\beta_2) (\psi_{t,2} u_{t,2})^2 \\ &\quad + \left\{ 2(\alpha_{t,2} + \beta_2) \left[r_{t,1} X_t - \sum_{i=1}^{n_L} (\alpha_{t,L_i} + \beta_{L_i}) \psi_{t,L_i} \right] \right. \\ &\quad \left. - 2 \sum_{i=1}^{n_L} D_{t,2L_i} \psi_{t,L_i} \right\} (\psi_{t,2} u_{t,2}) \\ &\quad + \left[r_{t,1} X_t - \sum_{i=1}^{n_L} (\alpha_{t,L_i} + \beta_{L_i}) \psi_{t,L_i} \right]^2 \\ &\quad + \sum_{i,j=1}^{n_L} (D_{t,L_i L_j} + D'_{L_i L_j} - \beta_{L_i} \beta_{L_j}) \psi_{t,L_i} \psi_{t,L_j}. \end{aligned} \quad (4.41)$$

Also, we can determine the expected price of asset i at time $t + \Delta t$ based on those attime t as

$$\mathbb{E}_{t, X_t, \psi_t} [\psi_{t+\Delta t, i}^u] = (r_{t, i} + \beta_i) \psi_{t, i}, \quad (4.42)$$

$$\mathbb{E}_{t, X_t, \psi_t} [\psi_{t+\Delta t, i} \psi_{t+\Delta t, j}] = \left(r_{t, i} r_{t, j} + \beta_i r_{t, j} + \beta_j r_{t, i} + D_{t, ij} + D'_{ij} \right) \psi_{t, i} \psi_{t, j}, \quad (4.43)$$

$$\mathbb{E}_{t, X_t, \psi_t} [X_{t+\Delta t} \psi_{t+\Delta t, i}] = \bar{a}_{t, i} X_t \psi_{t, i} + \bar{b}_{t, i} \psi_{t, i} + \sum_{j=1}^{n_L} \bar{c}_{t, ij} \psi_{t, i} \psi_{t, j}, \quad (4.44)$$

where

$$a_{t, i} = (r_{t, 1} + \alpha_{t, 2} C_t) (r_{t, L_i} + \beta_{L_i}) + D_{t, 2L_i} C_t + \beta_2 C_t r_{t, L_i} + D'_{2L_i} C_t, \quad (4.45)$$

$$b_{t, i} = \alpha_{t, 2} h_t (r_{t, L_i} + \beta_{L_i}) + D_{t, 2L_i} h_t + \beta_2 h_t r_{t, L_i} + D'_{2L_i} h_t, \quad (4.46)$$

$$\begin{aligned} c_{t, i} = & (\alpha_{t, 2} \eta_{t, j} - \alpha_{t, L_j}) (r_{t, L_i} + \beta_{L_i}) + (D_{t, 2L_i} \eta_{t, j} - D_{t, L_i L_j}) + (\beta_{t, 2} \eta_{t, j} - \beta_{t, L_j}) r_{t, L_i} \\ & + (D'_{t, 2L_i} \eta_{t, j} - D'_{t, L_i L_j}). \end{aligned} \quad (4.47)$$

We now prove Theorem 4.4.2 via the backward induction. It is trivial that equation (4.13)-(4.16) hold for $t = T$. Assuming equation (4.9) and equations (4.13)-(4.16) hold for $t \geq k + \Delta t$, we now check the case for $t = k$. Let $u = (u_k, u_{k+\Delta t}^*, \dots, u_T^*)$,

from definition (4.7), (4.8) and the Tower Property, we have

$$\begin{aligned}
f(k, X_k, \psi_k) &= \mathbb{E}_{k, X_k, \psi_k} [X_T^u] \\
&= \mathbb{E}_{k, X_k, \psi_k} \left[\mathbb{E}_{k+\Delta t, X_{k+\Delta t}^{u_k}, \psi_{k+\Delta t}^{u_k}} [X_T^{u*}] \right] \\
&= \mathbb{E}_{k, X_k, \psi_k} [g(k + \Delta t, X_{k+\Delta t}^{u_k}, \psi_{k+\Delta t}^{u_k})] \\
&= A_{k+\Delta t} \mathbb{E}_{k, X_k, \psi_k} [X_{k+\Delta t}^{u_k}] + \sum_{i=1}^{n_L} B_{k+\Delta t, i} \mathbb{E}_{k, X_k, \psi_k} [\psi_{k+\Delta t, i}] + M_{k+\Delta t} \\
&= A_{k+\Delta t} \left[[r_{k,1} + (\alpha_{k,2} + \beta_{k,2}) C_k] X_k + \sum_{i=1}^{n_L} [(\alpha_{k,2} + \beta_2) \eta_{k,i} \right. \\
&\quad \left. - (\alpha_{k,L_i} + \beta_{L_i})] \psi_{k,L_i} + (\alpha_{k,2} + \beta_{k,2}) h_k \right] \\
&\quad + \sum_{i=1}^{n_L} B_{k+\Delta t, i} (r_{k,L_i} + \beta_{L_i}) \psi_{k,L_i} + M_{k+\Delta t} \\
&= A_k X_k + \mathbf{B}_k^T \boldsymbol{\psi}_{k,L} + M_k,
\end{aligned}$$

$$\begin{aligned}
g(k, X_k, \psi_k) &= \mathbb{E}_{k, X_k, \psi_k} [(X_T^u)^2] \\
&= \mathbb{E}_{k, X_k, \psi_k} \left[\mathbb{E}_{k+\Delta t, X_{k+\Delta t}^{u_k}, \psi_{k+\Delta t}^{u_k}} [(X_T^{u*})^2] \right] \\
&= \mathbb{E}_{k, X_k, \psi_k} [f(k + \Delta t, X_{k+\Delta t}^{u_k}, \psi_{k+\Delta t}^{u_k})] \\
&= E_{k+\Delta t} \mathbb{E}_{k, X_k, \psi_k} [(X_{k+\Delta t}^{u_k})^2] + F_{k+\Delta t} \mathbb{E}_{k, X_k, \psi_k} [X_{k+\Delta t}^{u_k}] \\
&\quad + \sum_{i,j}^{n_L} G_{k+\Delta t, i,j} \mathbb{E}_{k, X_k, \psi_k} [\psi_{k+\Delta t, L_i} \psi_{k+\Delta t, L_j}] + \sum_{i=1}^{n_L} H_{k+\Delta t, i} \mathbb{E}_{k, X_k, \psi_k} [\psi_{k+\Delta t, L_i}] \\
&\quad + \sum_{i=1}^{n_L} I_{k+\Delta t, i} \mathbb{E}_{k, X_k, \psi_k} [X_{k+\Delta t}^{u_k} \psi_{k+\Delta t, L_i}] + N_{k+\Delta t} \\
&= E_k X_k^2 + F_k X_k + \boldsymbol{\psi}_k^T \mathbf{G}_k \boldsymbol{\psi}_k + \mathbf{H}_k^T \boldsymbol{\psi}_k + \mathbf{I}_k^T \boldsymbol{\psi}_k X_k + N_k.
\end{aligned}$$

Hence, equation (4.13) (4.14) holds for $t = k$.

Also, $E_{k+\Delta t} - A_{k+\Delta t}^2 \geq 0$ and $E_{k+\Delta t} > 0$ imply that

$$\begin{aligned}
E_k - A_k^2 &= (E_{k+\Delta t} - A_{k+\Delta t}^2) [r_{k,1} + (\alpha_{k,2} + \beta_2) C_k]^2 \\
&\quad + E_{k+\Delta t} [\sigma_{k,2}^2 \Delta t + (a_2^2 + b_2^2) \lambda_2 \Delta t] C_k^2 \\
&\geq 0
\end{aligned} \tag{4.48}$$

$$\begin{aligned}
E_k &= E_{k+\Delta t} \left\{ [\sigma_{k,2}^2 \Delta t + (a_2^2 + b_2^2) \lambda_2 \Delta t] C_k^2 + [r_{k,1} + (\alpha_{k,2} + \beta_2) C_k]^2 \right\} \\
&> 0
\end{aligned} \tag{4.49}$$

Therefore, equation (4.15) and equation (4.16) hold for $t = k$. This finished the proof for Theorem 4.4.2.

5 Simulation and Conclusion

In this chapter, we focus on Mean Variance Asset-Liability selection problem in section 3.1.4. Numerical comparisons and discussion on the performance for different control constraints were exhibited.

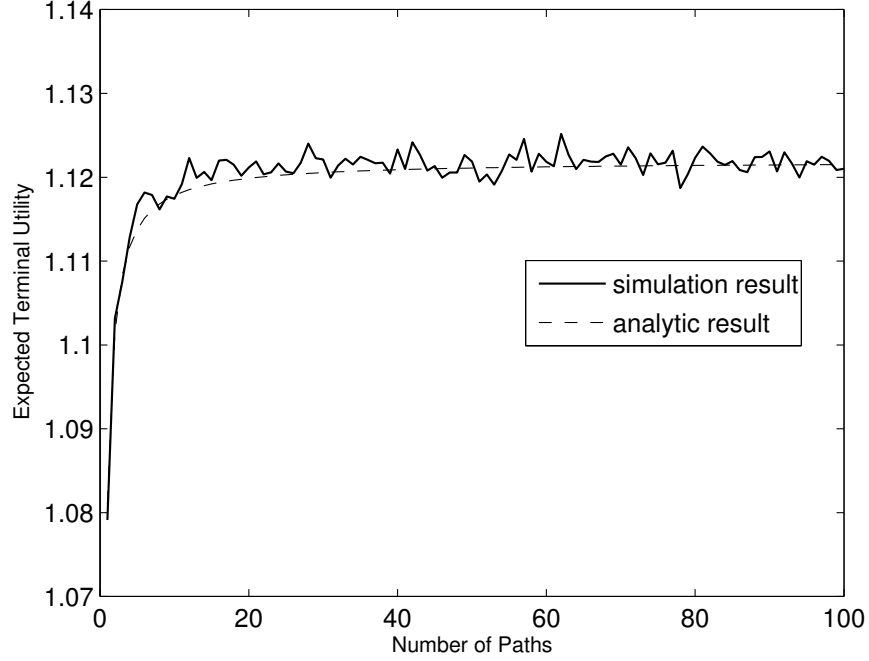


Figure 5.0.1: Comparison of Simulation and Analytic Result

In figure 5.0.1, we compare analytic solution in equation (4.9) and the simulation result. The simulation result (as shown in solid line) lies closely around the analytic solution (as shown in dotted line). We use the Mean Squared Error (MSE) to measure the difference between results from our explicit solution and simulation result. Figure 5.0.2 and 5.0.3 plots the Mean Squared Error (MSE) of the utility up to 10^5 number of paths. The mean squared difference decreases, in both single period and multi period cases. This shows that the difference between the estimator of our analytic result and the value from simulation is rather small and tends to zero as the number of simulation paths increases.

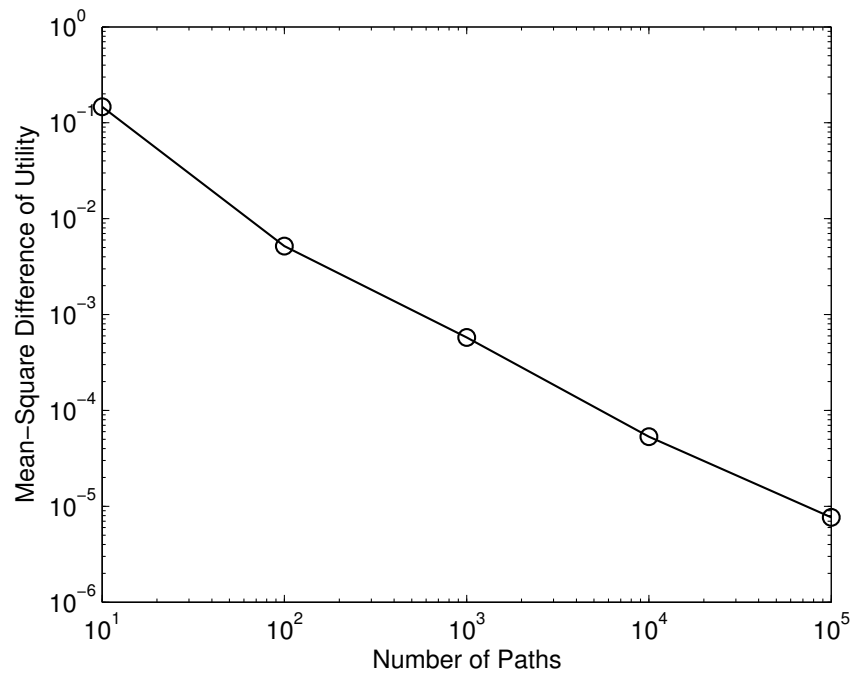


Figure 5.0.2: Mean-Squared Error in One-Step Model

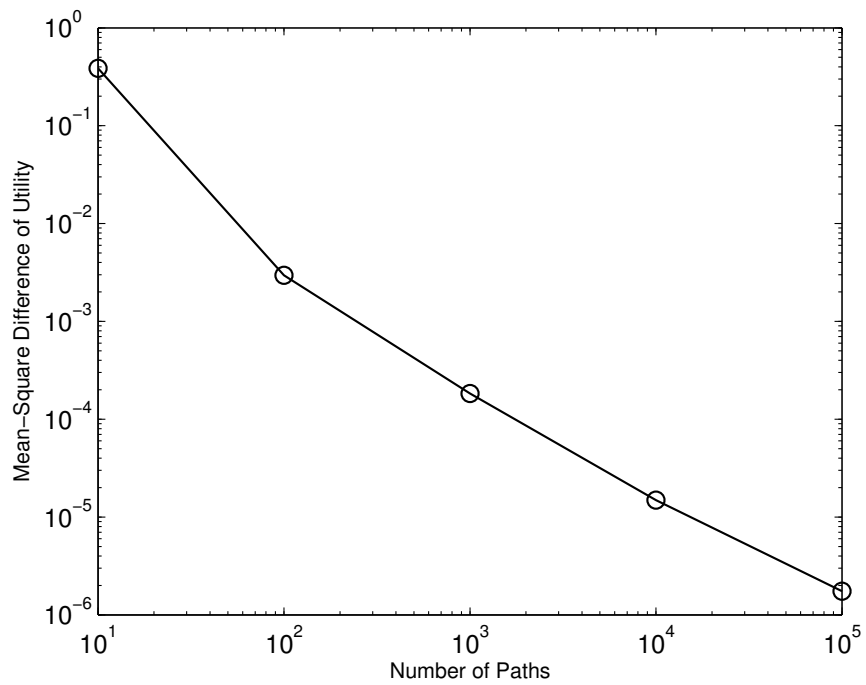


Figure 5.0.3: Mean-Squared Error in Multi-Step Model

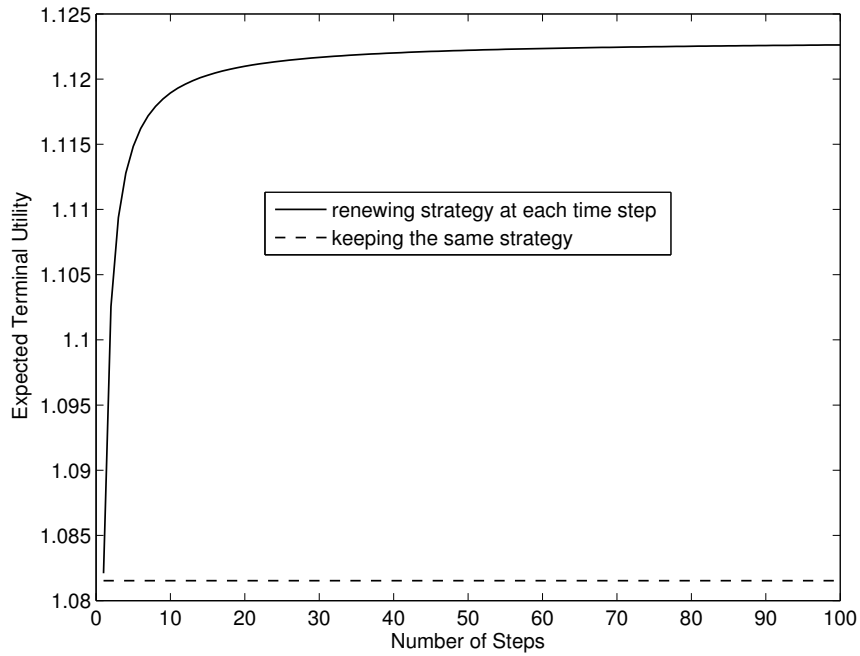


Figure 5.0.4: Comparison of Trading Strategies

Figure 5.0.4 shows a comparison between two trading methods. Black solid line represents the method of renewing each investment strategy at each time slot, while the dashed line represents the method of determining the investment strategy at the beginning and keeping the same investment throughout the whole time horizon. This figure shows that keeping investment strategies at each time slot has better expected terminal utility.

Figure 5.0.5 shows how the expected terminal utility behaves against different risk averse coefficients. Figure 5.0.6 shows how the expected terminal utility behaves against investment in risky asset. Investors tend to invest more in risky asset when risk averse is small. As the risk averse coefficient grows larger, investors tend to borrow less from riskless asset to invest in the risky asset.

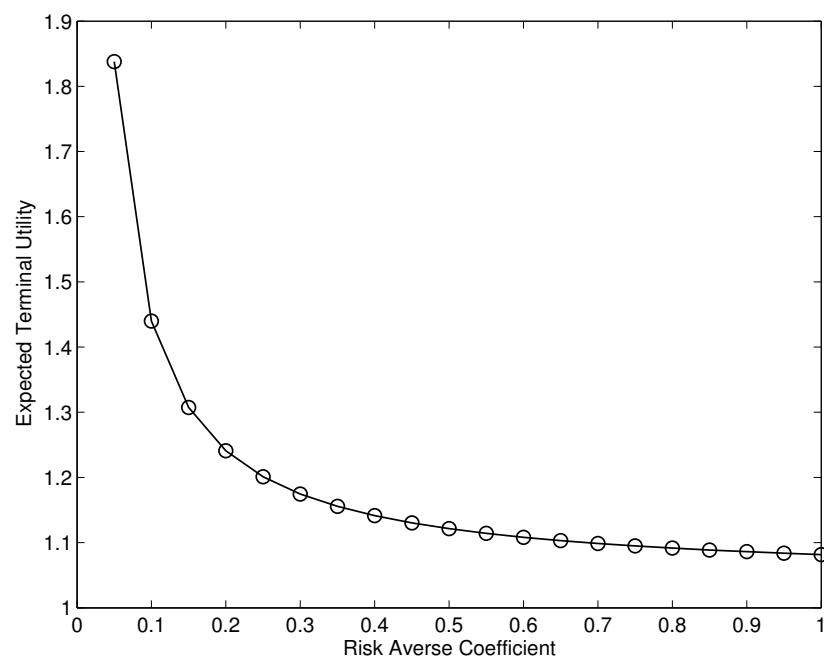


Figure 5.0.5: Expected Terminal Utility to Risk Averse Coefficient

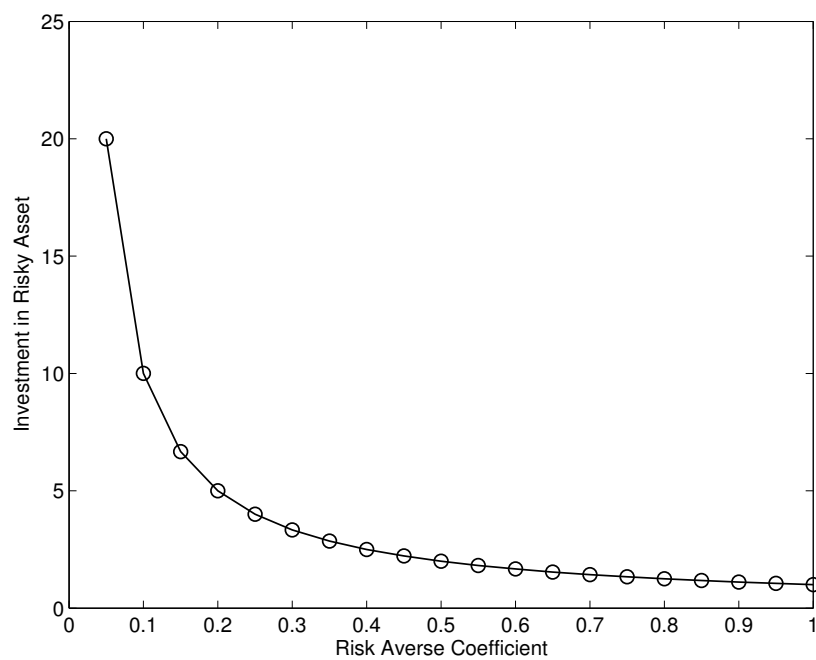


Figure 5.0.6: Investment in Risky Asset to Risk Averse Coefficient

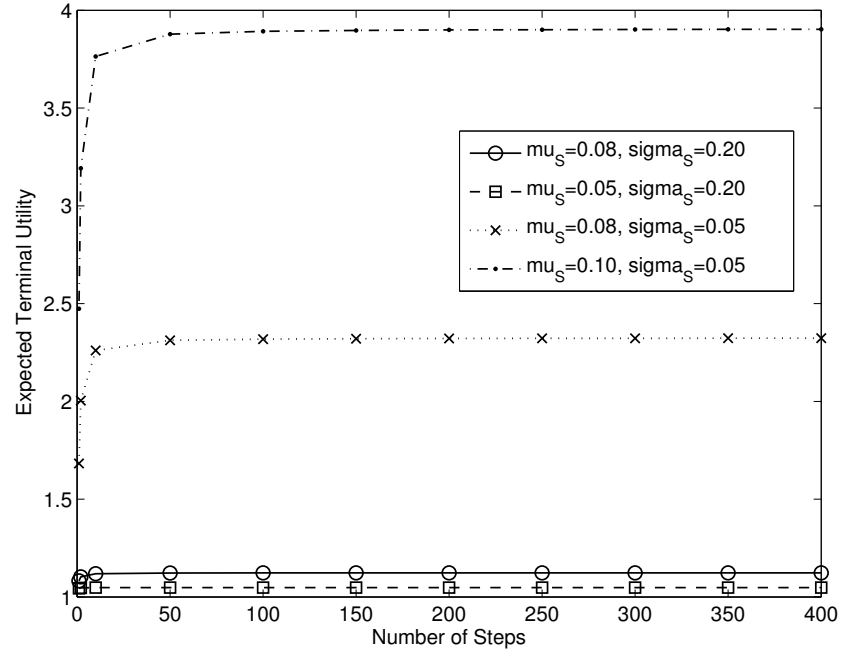


Figure 5.0.7: Comparing among different parameters (Risky Asset)

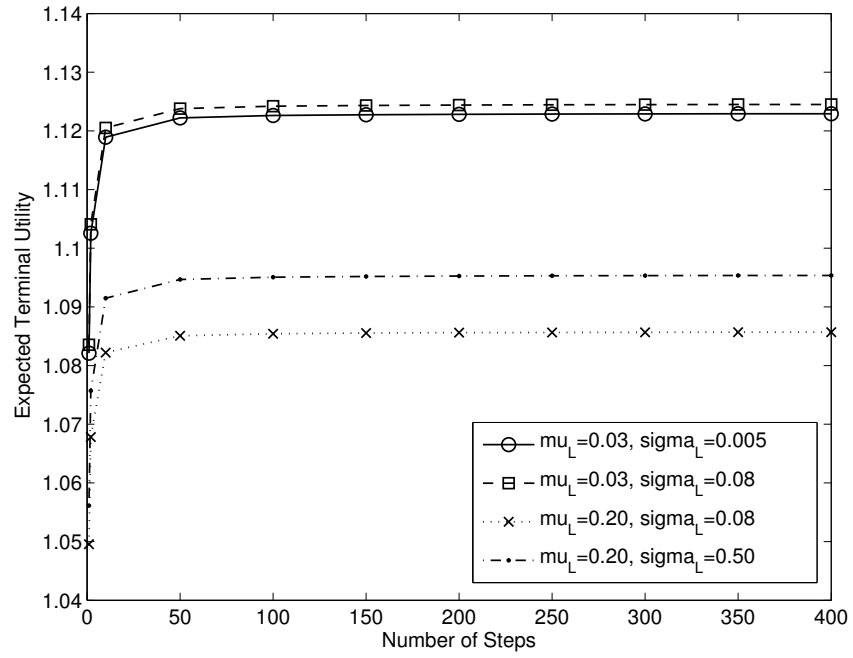


Figure 5.0.8: Comparing among different parameters (Liability)

Figure 5.0.7 and figure 5.0.8 exhibit the comparison among four pairs of parameters of both risky asset and liability. In figure 5.0.7, we have four different pairs of parameters for appreciate-rate and volatility of risky assets. The figure show that the expected terminal utility has a quiet significant increment as the appreciate-rate increases or the volatility decreases. However, in figure 5.0.8, the case of liability yields an opposite result. Expected terminal utility decreases as the appreciate-rate decreases, and expected terminal utility decreases as the volatility increases. Moreover, the fluctuation in expected terminal utility is not that significant.

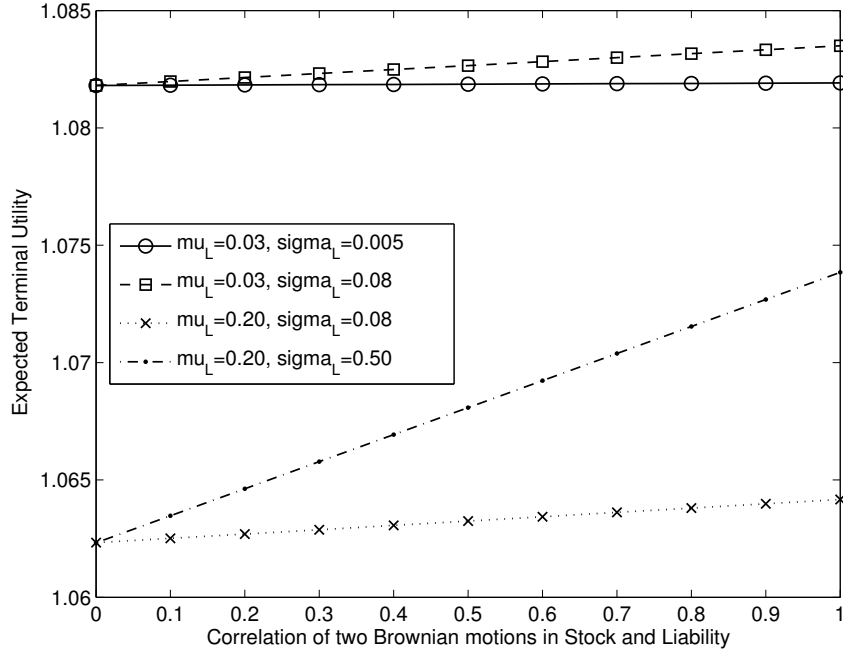


Figure 5.0.9: Expected Terminal Utility for different correlations of two Brownian motions in risky asset and liability

Figure 5.0.9 shows the effect from the correlation of two Brownian motions in risky asset and liability. For fixed parameters of risky assets, we choose four pairs for parameters of liability and vary the coefficient of correlation. We can see that the change of expected terminal utility is not obvious with a variation of less than 1%.

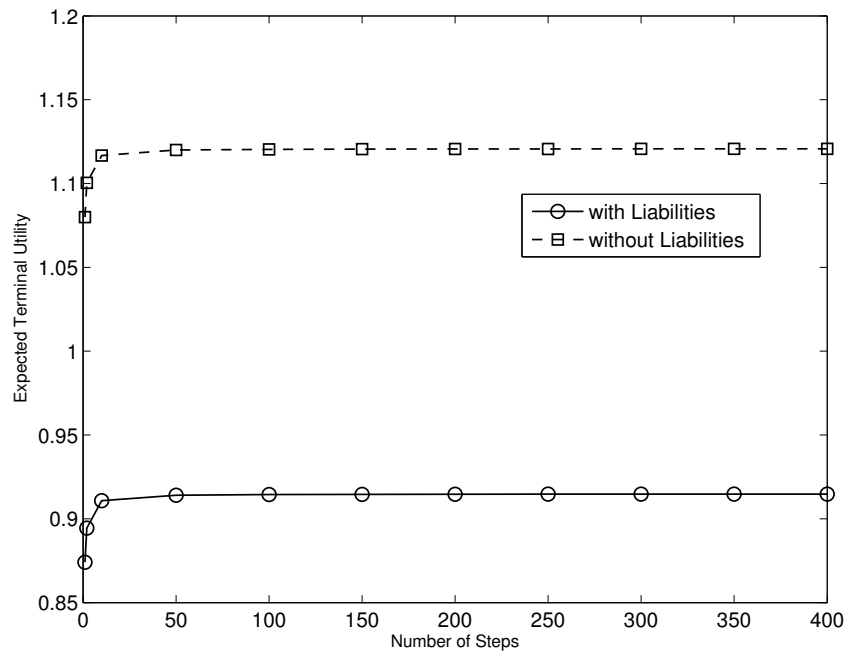


Figure 5.0.10: Comparing Expected Terminal Utility with and without Liabilities

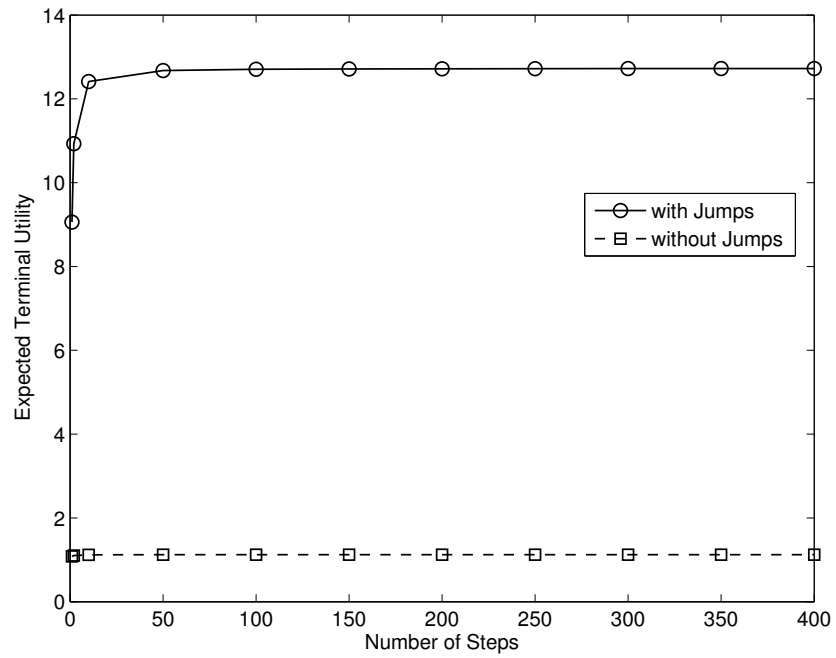


Figure 5.0.11: Comparing Expected Terminal Utility with and without Jumps

Figure 5.0.10 compares the expected terminal wealth with and without the liability process. The case without liabilities returns a higher expected terminal utility, while the case with liability results in a lower expected terminal utility. An investor starts with holding certain amount of risk-free asset and risky assets, will push down one's total wealth when holding liabilities. This leads to a lower expected terminal utility.

Figure 5.0.11 shows how the jump process contributes to the expected utility. The case with jump results in a higher utility compared to the case without, that means that one has the possibility of receiving more return when jumps present. This highlights the fact that liability has a significant impact on asset allocation for Asset-Liability management, and we need to care for the liability process and take jump process into account.

6 Conclusions

In this thesis, we discuss the Asset-Liability portfolio selection problem in a multi-period portfolio selection problem. Under the mean-variance framework, we obtain the optimal investment strategy both theoretically and numerically. Our results show why it is important to take liabilities and market jumps into account.

The main limitation of our model is that we have only one risky asset. If we have multiple risky assets, we have shown that our multi-step problem would be analytically intractable. One would have to resort to numerical methods.

Our analytic model may be extended to the case for a network of investors with linkage liabilities. This would be one of our future directions.

7 Appendix

7.1 Matlab Code for Analytic Solution

7.1.1 Matlab Code for Analytic Wealth

```
function [ wealth,investment,Stock,Liability ] ...
    = AnalyticWealth( problem,nSteps,gamma)

[ ~,~,~,~,~,~,~,~,~,C,eta,h,aalpha_S,aalpha_L,...
    r_b,r_S,r_L,beta_S,beta_L,~,~,~,~,~,~ ]...
    = AnalyticCoefficient( problem ,nSteps ,gamma);

i = 1;
wealth(1,i) = problem.initialwealth;
Stock(1,i) = problem.initialstockprice;
Liability(1,i) = problem.initialliability;
investment(1,i) = (C(1,i)*wealth(1,i)+eta(1,i)*Liability(1,i)+h(1,i));

for i = 2:nSteps
    wealth(1,i) = r_b * wealth(1,i-1)...
        + (aalpha_S+beta_S) * investment(1,i-1) ...
        - (aalpha_L + beta_L) * Liability(1,i-1);
    Stock(1,i) = (r_S + beta_S)*Stock(1,i-1);
    Liability(1,i) = (r_L + beta_L) *Liability(1,i-1);
    investment(1,i) = (C(1,i)*wealth(1,i)...
        +eta(1,i)*Liability(1,i)+h(1,i));
end

i=nSteps+1;
wealth(1,i) = r_b * wealth(1,i-1) ...
    + (aalpha_S+beta_S) * investment(1,i-1) ...
```

```
- (aalpha_L + beta_L) * Liability(1,i-1);
```

```
end
```

7.1.2 Matlab Code for Analytic Objective

```
function [MeanofWealth, MeanofWealthSquare,utility,investment ] ...
    = AnalyticObjective( problem,nSteps,gamma)

[ wealth,investment,~,Liability ] ...
    = AnalyticWealth( problem,nSteps,gamma);
[ A,B,M,E,F,G,H,I,N,C,eta,h,aalpha_S,aalpha_L,r_b,~,r_L,...
    beta_S,beta_L,D_SL,D2_SS,D2_LL,a2,b2,c2 ]...
    = AnalyticCoefficient( problem ,nSteps,gamma );

sigma_S = problem.sigma_S;
sigma_L = problem.sigma_L;
dt = problem.T/nSteps;
MeanofWealth = zeros(1,nSteps);
MeanofWealthSquare = zeros(1,nSteps);
VarianceofWealth = zeros(1,nSteps);

for i = 1:nSteps
    MeanofWealth(1,i) = A(1,i+1)* ((aalpha_S+beta_S)*investment(1,i)...
        +r_b*wealth(1,i)-(aalpha_L+beta_L)*Liability(1,i))...
        +B(1,i+1)*(r_L+beta_L)*Liability(1,i)+M(1,i+1);
    MeanofWealthSquare(1,i) = ...
        E(1,i+1)*((sigma_S^2*dt+D2_SS+aalpha_S^2+2*aalpha_S*beta_S)...
            *(investment(1,i)^2)+ (2*(aalpha_S+beta_S)*(r_b*wealth(1,i)...
            -(aalpha_L+beta_L)*Liability(1,i))...
            -2*D_SL*Liability(1,i))*investment(1,i)...
```



```

+ ((r_b*wealth(1,i)-(aalpha_L+beta_L)*Liability(1,i))^2)...
+ (sigma_L^2*dt+D2_LL -beta_L^2)*Liability(1,i)^2)...
+F(1,i+1)*((r_b+(aalpha_S+beta_S)*C(1,i))* wealth(1,i) ...
+ ((aalpha_S + beta_S)*eta(1,i) - aalpha_L - beta_L)...
*Liability(1,i)+( aalpha_S+beta_S)*h(1,i))...
+(G(1,i+1)*((r_L^2+2*r_L*beta_L+sigma_L^2*dt+D2_LL)...
*Liability(1,i)^2)))+(H(1,i+1)*(r_L+beta_L)*Liability(1,i))...
+I(1,i+1)*(a2(1,i)*wealth(1,i)*Liability(1,i)+b2(1,i)...
*Liability(1,i)+c2(1,i)*Liability(1,i)^2)+N(1,i+1);
VarianceofWealth(1,i) = MeanofWealthSquare(1,i)...
- (MeanofWealth(1,i))^2;
end

utility = MeanofWealth -0.5*gamma*VarianceofWealth;

end

```

7.1.3 Matlab Code for Plotting Mean Squared Error

```

function [utility,analyticutility]=PlotErrorOneStepProblem( )

points = 1:5;
nPaths = 10.^points;
problem = CreateParametersOneStepProblem();
niterations = 100;
error = zeros(length(nPaths),length(niterations));
[ ~,~,analyticutility,~ ] ...
    = AnalyticObjective(problem,problem.nSteps,problem.gamma);
%analytic results
for j = 1:niterations

```

```

for i = 1:length(nPaths)
    initialwealth = problem.initialwealth;
    initialstockprice = problem.initialstockprice;
    initialliability = problem.initialliability;
    dt = problem.dt;
    [ ~,utility ] = OptimisebyFmincon( problem,nPaths(i),...
        initialwealth,initialstockprice,initialliability,dt );
    %simulate by Fmincon fct
    error(i,j) = abs(utility + analyticutility(1,problem.nSteps));
end

end

MeanSquareError = mean(error.^2,2);
loglog(nPaths,MeanSquareError,'-ko',...
    'MarkerSize',8,...
    'LineWidth',1);
xlabel('Number of Paths');
ylabel('Mean-Square Difference of Utility');
set(gca,'fontSize',12);

end

```

7.1.4 Matlab Code for Plotting Comparison of Utility

```

function [ ] = PlotUtilityComparison( )

[ problem1 ] = CreateParametersMultiStepProblemComparison( );
[ problem2 ] = CreateParametersMultiStepProblemComparison2( );
[ problem3 ] = CreateParametersMultiStepProblemComparison3( );
[ problem4 ] = CreateParametersMultiStepProblemComparison4( );

```

```

nSteps = problem1.nSteps;
rho = 0:0.1:1;
gamma = problem1.gamma;

case1 = ones(length(rho));
case2 = ones(length(rho));
case3 = ones(length(rho));
case4 = ones(length(rho));

for i = 1: length(rho)
    [ ~,~,utility1 ]...
        = AnalyticObjectiveComparison( problem1,nSteps,gamma,rho(i) );
    [ ~,~,utility2 ]...
        = AnalyticObjectiveComparison( problem2,nSteps,gamma,rho(i) );
    [ ~,~,utility3 ]...
        = AnalyticObjectiveComparison( problem3,nSteps,gamma,rho(i) );
    [ ~,~,utility4 ]...
        = AnalyticObjectiveComparison( problem4,nSteps,gamma,rho(i) );

    case1(i) = utility1(1,nSteps);
    case2(i) = utility2(1,nSteps);
    case3(i) = utility3(1,nSteps);
    case4(i) = utility4(1,nSteps);

end

plot(rho,case1,'-ko',...
     rho,case2,'--ks',...
     rho,case3,':kx',...
     rho,case4,'-.k.',...
     'MarkerSize',8,...
     'LineWidth',1)
legend('mu_L=0.03, sigma_L=0.005','mu_L=0.03, sigma_L=0.08',...
      'mu_L=0.20, sigma_L=0.08','mu_L=0.20, sigma_L=0.50',0);

```

```

hold off
xlabel('Correlation of two Brownian motions in Stock and Liability');
ylabel('Expected Terminal Utility');
set(gca,'fontsize',12);

end

```

7.2 Matlab Code for Simulation

7.2.1 Matlab Code for Simulating Stock and Liability

```

function [ Stock,Liability ] = SimulateStockandLiability( problem,...
    nPaths,nSteps,initialstockprice,initialliability,dt )

mu_L = problem.mu_L;
sigma_L = problem.sigma_L;
rho_SL = problem.rho_SL;
mu_S = problem.mu_S;
sigma_S = problem.sigma_S;

[ dW1,dW2 ] ...
    = GenerateBrownianMotion( nPaths,nSteps );
[ StockJump,LiabilityJump ] ...
    = SimulateJumps( problem,nPaths,nSteps );

S0 = initialstockprice;
S1 = (1+ mu_S*dt)...
    +sigma_S*sqrt(dt)*dW1...
    +StockJump;
Stock = S0*cumprod(S1,2);

L0 = initialliability;

```

```

L1 = (1+ mu_L*dt)...
      +sigma_L*rho_SL*sqrt(dt)*dW1...
      +sigma_L*sqrt(1-rho_SL^2)*sqrt(dt)*dW2...
      +LiabilityJump;
Liability = L0*cumprod(L1,2);

end

```

7.2.2 Matlab Code for Simulating Wealth

```

function [ Wealth ] = SimulateWealth( problem,investment,dt,...
    initialwealth,initialstockprice,initialliability,Stock,Liability )

u0 = investment;
x0 = initialwealth;
S0 = initialstockprice;
L0 = initialliability;

r= problem.r;
G_b = 1+r*dt;
initialbond = x0 + L0 - u0;
WealthinBond = G_b*initialbond;
WealthinStock = u0/S0*Stock;
Asset = WealthinBond + WealthinStock;
Wealth = Asset - Liability;

end

```

7.2.3 Matlab Code for Computing Utility Function

```

function [ Utility ] = ComputeUtility( problem,investment,dt,...

```

```

        initialwealth,initialstockprice,initialliability,Stock,Liability )

gamma = problem.gamma;

[ Wealth ] = SimulateWealth( problem,investment,dt,initialwealth,...
        initialstockprice,initialliability,Stock,Liability );

ExpectationofWealth = mean(Wealth);
VarianceofWealth = var(Wealth);
Utility = ExpectationofWealth - 0.5*gamma*VarianceofWealth;

end

```

7.2.4 Matlab Code for Fmincon Optimisation

```

function [ investment,fval ] = OptimisebyFmincon( problem,nPaths,...
        initialwealth,initialstockprice,initialliability ,dt)

nSteps = 1;

[ Stock,Liability ] = SimulateStockandLiability( problem,nPaths,...
        nSteps,initialstockprice,initialliability,dt );

function disutility = Objective(u)
    utility = ComputeUtility( problem,u,dt,initialwealth,...
        initialstockprice,initialliability,Stock,Liability );
    disutility = -utility;

end

options=optimset('fmincon');
options=optimset(options,'Display','off');
[investment,fval,exitflag]=fmincon(@Objective,1.0,[],[],[],[],...
        -Inf,Inf,[],options);
assert(exitflag>0);

end

```

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